A FROBENIUS AND HOPF ALGEBRAIC FORMULATION OF QUANTUM FIELD THEORY

A Thesis
Submitted to the Faculty of Graduate Studies and Research
In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy
In
Mathematics
University of Regina

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Regina, Saskatchewan
December 2025

Abstract

Hopf algebras and Frobenius algebras are two kinds of associative algebras that appear naturally in several fields of mathematics and physics. While Frobenius algebras tend to encode topological, geometric, and analytic structure, Hopf algebras can be used to encode combinatorial or algebraic structure. In this work we pursue one notion of infinite-dimensional Frobenius algebra and define a kind of module structure of Hopf algebras acting on Frobenius algebras, called Hopf-Frobenius modules.

Hopf-Frobenius modules are shown to provide a framework for unifying various constructions of quantum field theory, and are also used to prove a version of the Lie correspondence between Lie groups and Lie algebras.

We define a notion of Hopf-Frobenius quantum field theory locally, and provide a refinement of algebraic quantum field theory which relies on Hopf algebras with Laplace pairings.

Keywords: Hopf algebras, Frobenius algebras, quantum field theory, algebraic quantum field theory.

Acknowledgements

This work would not have been possible without the guidance and support of members throughout the University of Regina community. My heartfelt thanks to Allen Herman, Karen Meagher, Francis Bischoff, Sarah Carnochan Naqvi, Shaun Fallat, Leslie Robbins, Remus Floricel, Gojko Vujanovic, Patrick Maidorn, Robert Petry, Fernando Szechtman, and Philip Charrier.

My overwhelming gratitude goes to the members of my thesis committee: Donald Stanley, Martín Argerami, and Pierre Ouimet, and above all, to my thesis advisor, Martin Frankland.

This work was supported by an NSERC Postgraduate Scholarship.

Post-Defense Acknowledgements

My sincerest thanks also to my External Examiner, Donald Yau.

Dedication

To my earliest friends —
    To JAK, to Joe, to Zander
To my first teachers —
    To Lance, to Mr. Caspick, to Tracey
To all of those from Bayview,
    Who expanded what I knew as home
To every member of my family
    That shapes the world around me

But more than words can say —
    To Mom, to Dad, to Lauren, to Arthur.

Transparency Statement

I declare no AI-assisted technology was used in the preparation of this thesis.

Contents

Chapter 1

Introduction

1.1 Background

The field of mathematical physics admits a great difficulty in that its researchers (viz. mathematicians and physicists) typically understand their field using very different languages, with different motivations, techniques, conjectures, and conventions. The question repeatedly arises, how should one engage with the field in order to best play to the strengths of both styles of researcher?

While mathematicians can construct rigorous frameworks and tools for studying physics, and can prove results in great generality, this comes with a serious caveat: once a mathematician has produced a large class of structures which should mathematically capture certain physical systems, it is usually very unclear, from a mathematician's perspective, which of these mathematically viable options should correspond to physical reality. On the other hand, physicists can apply intuition and experiment in order to pinpoint the physically relevant features of mathematical models, but physicists' work can lack mathematical rigour and the techniques and tools they have available are limited by the areas of mathematics with which they have the most dexterity.

In quantum field theory in particular, a wide variety of approaches have developed for studying field theories. These approaches generally come in three flavours:

1. $\boxed{\text{Detailed:}}$ Axiomatic approaches to quantum field theory, like those of Wightman [35] and the work of Osterwalder-Schrader [28] try to set down what quantum field theory *is* or should be. They involve lengthy lists of axioms which, when satisfied, allow one to prove that a given field theory has certain desirable physical properties. However, finding non-trivial four-dimensional examples which satisfy these axioms remains an open problem.

2. $\boxed{\text{General:}}$ On the other hand, approaches like those of algebraic quantum field theory (AQFT) [16] and (extended) topological quantum field theory (TQFT) [5] study physics from the point of view of mathematics which should somehow capture the minimal amount of structure necessary to reasonably call something a quantum field theory. This provides a very flexible and general theory, however finding physically interesting theories within the huge ocean of viable AQFTs and TQFTs is tremendously challenging.

3. $\boxed{\text{Specific:}}$ Areas like the study of vertex operator algebras [15, 8] restrict themselves to considering certain types of field theories (for example conformal ones) in order to have something concrete to analyse, without having to commit themselves to defining what a quantum field theory should be in general. Of course, this can often leave the

question of the relationships between different types of field theories rather mysterious.

In this thesis our goal is to present a new mathematical structure (Hopf-Frobenius modules) which cuts to the heart of what a given quantum field theory *does* for the physicists who study it, rather than trying to set out what a quantum field theory *is* philosophically or mathematically. In other words, when a physicist writes down a mathematical framework which they call a 'quantum field theory', it is with the goal of computing some prediction about a quantum system, which can then be tested experimentally. For a physicist then, an ideal mathematical notion of a quantum field theory should be something which has structure meant to capture or produce *measurable features* of the universe.

With this in mind, the development of Hopf-Frobenius modules is centered around the concept of **correlation functions** or **correlators** (the primary experimentally measurable concept in quantum field theory). It is our hope that by putting measurement at the center of our formalism of quantum field theory, the mathematics set out in this work can be of use to physicists who can refine the Hopf-Frobenius set-up to best pertain to their systems of interest.

1.2 Outline of Contributions

The two algebraic structures of interest in this thesis (Frobenius algebras and Hopf algebras) have been studied individually at length. However in this work we introduce generalizations of these algebras in order to capture structure of physical interest.

$\boxed{\text{Frobenius algebras:}}$ The standard notion of Frobenius algebra assumes the algebras to be finite-dimensional [32, 20]. With this assumption, Frobenius algebras can be defined equivalently as algebras with a certain kind of non-degenerate bilinear form, or as algebras which are also coalgebras that satisfy certain compatibility conditions between the coalgebraic and algebraic structure. One approach to discussing infinite-dimensional Frobenius algebras is to drop the requirement that Frobenius algebras have a counit, while keeping the comultiplication [4, 1]. From our perspective however, the counit (or more generally the bilinear form) plays the key role of defining correlators for Frobenius algebras. As such, we pursue a notion of Frobenius algebra which drops the comultiplication requirement instead.

$\boxed{\text{Weak associativity:}}$ This generalized notion of Frobenius algebra also is given a relaxed associativity condition: instead of the algebra itself being associative, we only require that it appears associative under the bilinear form

$$\langle xy, z \rangle = \langle x, yz \rangle \, .$$

If we think of the pairing in physical terms, as something meant to encode experimentally verifiable information, this says that our Frobenius algebra only needs to *appear associative experimentally* — whether the algebra actually is associative or not isn't necessarily of concern. This will be a common theme. For example, in our definition of Hopf-Frobenius module, certain defining conditions will only be required to be satisfied inside of a particular bilinear form.

$\boxed{\text{Hopf and bialgebras:}}$ Hopf algebras (and their more general relative, bialgebras) have long been studied in both finite and infinite dimensions [36, 30, 12]. In [13] Fauser showed that Wick's theorem (a combinatorial result about computing correlators in quantum field theory) follows from the structure of a Laplace pairing on a Hopf or bialgebra. Wick's theorem as proved in [13] holds for bosons and fermions. In [9] Brouder likewise restricts to the case of Wick's theorem for bosons. We extend these results by defining generalized bialgebras called Q-bialgebras, whose commutation relations can incorporate more exotic terms than the ± 1 of commutators/anticommutators. Our generalized Wick's theorem is in Theorem 3.4.6 and Proposition 3.4.13. This structure is physically motivated by the types of commutation relations that appear for anyons and the canonical commutation relations.

Creation/annihilation operators: Joni and Rota introduced a notion of creation and annihilation operators that can be defined for any coalgebra [19]. We demonstrate that for an appropriate choice of Hopf algebra and Laplace pairing, the Joni–Rota version of creation/annihilation operators matches with that of the standard notion from quantum field theory (see Example 5.1.3).

Hopf-Frobenius modules: We introduce the new notion of Hopf-Frobenius modules (Definition 4.1.1) and show how it captures some of the essential structure leveraged by physicists working with tools from geometry, algebra, and analysis.

Lie correspondence: The classical Lie correspondence between Lie groups and Lie algebras is a deep connection between the differential-geometric concept of Lie groups, and the algebraic structure in Lie algebras. We show how the classical Lie groups and Lie algebras can be viewed as Hopf-Frobenius modules, and show that every Hopf-Frobenius module comes with both a corresponding group and Lie algebra (Lemma 4.3.2 and Proposition 4.3.7). We then demonstrate that in the case of the classical Lie groups/algebras, viewed as Hopf-Frobenius modules, a version of the Lie correspondence can be proven which associates the appropriate Lie groups and Lie algebras to each other (Proposition 4.3.5, Proposition 4.3.6, and Proposition 4.3.16). We also sketch how the correspondence can be proven in the case of the exceptional Lie groups/algebras (Example 4.4.1).

$\boxed{\text{Hopf quantization:}}$ Deformation quantization is a mathematically rigorous way of expressing the quantization process of quantum physics. We introduce a notion of quantization for any Hopf-Frobenius module, which produces a new quantized Hopf algebra, and show how this recovers ordinary deformation quantization for an appropriate choice of Hopf-Frobenius module (Example 5.2.1).

$\boxed{\text{Hopf-Frobenius QFT:}}$ We define Hopf-Frobenius quantum field theories (Definition 5.3.1), and show how many of the standard examples of field theories can be formulated in this framework(see Section 5.3.4, Section 5.3.5, and Section 5.4). Additionally, we present a version of the Wightman axioms which re-expresses the usual axioms in terms of the structure of Hopf-Frobenius modules (see Section 5.5).

$\boxed{\text{(Pr)operads:}}$ Given any operad $\mathcal{O}$ and small category $\mathcal{C}$, we describe an operad $\mathcal{O}^{\mathcal{C}}$ whose algebras are the functors from $\mathcal{C}$ to the category of $\mathcal{O}$-algebras, a construction left as an exercise in [41, Chapter 14] (Proposition 6.3.2). We then prove an analogous theorem for properads (Proposition 6.4.7). The version of the theorem for the associative operad was used in the construction of the AQFT operad of Benini, Schenkel, and Woike [6]. In their version of AQFT, there is no explicit structure which encodes correlation functions for the theory. We use the properadic version of the theorem in order to define a properad of 'Laplace Hopf AQFTs', whose algebras are AQFTs in which the algebras of observables are bialgebras with Laplace pairings (and thus have the structure of correlation functions); see Proposition 6.5.6.

1.3 Outline of Thesis

In Chapter 2 we adopt a generalization of Frobenius algebras and recall some basic properties of Frobenius algebras, showing that they hold for this generalization. We then provide an extensive list of examples of generalized Frobenius algebras, showing how this notion of Frobenius algebra captures a particular structure that reappears throughout analysis, geometry, and algebra.

Chapter 3 investigates Hopf and bialgebras and introduces the generalization of Q-bialgebras, which allows us to state and prove a more general version of Wick's theorem.

In Chapter 4 we bring Hopf and Frobenius algebras together in what we call *Hopf-Frobenius modules*. We formulate and prove a version of the Lie correspondence which holds for Hopf-Frobenius modules.

Chapter 5 is an exploration of how Hopf-Frobenius modules can be used to describe results and notions from physics.

Chapter 6 studies operads and properads, leading to the construction of a properad of Hopf-Laplace quantum field theories.

Chapter 2

Frobenius Algebras

In our formulation of quantum field theory, Frobenius algebras will play the role of algebras of states and fields, as well as the algebras encoding the geometry of spacetime. This flexibility to encode both analytic and geometric structure will provide a powerful framework for unifying seemingly unrelated structure in physics.

Classically, Frobenius algebras have been explicitly defined to be finite-dimensional. With the motivation of encoding analytic structure like algebras of functions, we will need a more general notion of Frobenius algebra, which we define below.

2.1 Infinite-dimensional Frobenius Algebras

Definition 2.1.1: $*$-ring

A **$*$-ring** is a ring R with an involution $* : R \to R$ such that

$$(x + y)^* = x^* + y^*$$
$$(xy)^* = y^* x^*$$
$$1^* = 1,$$

for all $x, y \in R$.

Note: Every commutative ring is a $*$-ring with trivial involution.

Convention: In what follows we will consider k to be a $*$-field unless otherwise specified.

Definition 2.1.2: Graded $*$-Algebra

A **graded $*$-algebra** is a graded k-algebra A with an involution $* : A \to A$ such that

$$(\alpha x + \beta y)^* = \alpha^* x^* + \beta^* y^*$$
$$(xy)^* = x^* y^*,$$

for all $x, y \in A$ and $\alpha, \beta \in k$.

Convention: The involution in a $*$-algebra is often required to satisfy

$(xy)^* = y^* x^*$. However, in order to treat ordinary algebras (over ordinary, non-starred rings) and $*$-algebras on the same footing, we will replace this requirement by multiplicativity of the involution.

Example 2.1.3: Graded $*$-Algebra: Graded Algebra

Every graded algebra A over an ordinary (trivially starred) commutative ring has a trivial involution $x^* = x$ for all $x \in A$. Since every algebra can be given a trivial grading, it follows that every algebra over a trivially starred ring can be given the structure of a $*$-algebra according to our definition.

2.1.1 Topology on Frobenius Algebras

We next introduce a few topological preliminaries. The fact that the non-degenerate pairing that appears in the definition of Frobenius algebras induces a topological structure will prove key to encoding geometric structure.

Definition 2.1.4: Dual Pairing of Vector Spaces

A **dual pairing** of vector spaces, $(V, W, \langle\cdot,\cdot\rangle)$, consists of k-vector spaces V and W, and a non-degenerate bilinear form

$$\langle\cdot,\cdot\rangle : V \otimes W \to k. \tag{2.1}$$

$\boxed{\text{Intuition:}}$ Since the bilinear form is non-degenerate, we have injections $V \to W^*$ and $W \to V^*$, given by $v \to \langle v, - \rangle$ and $w \to \langle -, w \rangle$, respectively. Because of this, we would like to think of V as a *nice version* of the dual of W (and vice versa).

Definition 2.1.5: Weak Topology

Let $(V, W, \langle \cdot, \cdot \rangle)$ be a dual pairing. The **weak topology on V** with respect to the dual pairing is the weakest topology such that

$$\langle -, w \rangle : V \to k$$

is continuous for all $w \in W$. Similarly, the **weak topology on W** with respect to the pairing is the weakest topology such that each

$$\langle v, - \rangle : W \to k$$

is continuous.

$\boxed{\text{Convention:}}$ When we have a dual pairing of a space with itself

$$\langle \cdot, \cdot \rangle : V \otimes V \to k,$$

we will take the **weak topology on V** to mean the weakest topology such that both

$$\langle -, v \rangle, \langle v, - \rangle : V \to k$$

are continuous for all $v \in V$.

Lemma 2.1.6

Let V be a vector space, and $\{(W_i, \tau_i)\}_{i \in I}$ a family of topological vector spaces. For any collection of linear maps $\{f_i : V \to W_i\}_{i \in I}$, the initial topology with respect to those maps makes V into a topological vector space.

Furthermore, if each $\{(W_i, \tau_i)\}_{i \in I}$ is locally convex, then the initial topology on V is locally convex as well.

Proof. See [37, Theorem 1.5] and [37, Example 2.5(b)]. □

Underlying field: In what follows we shall assume our Frobenius algebras are over the field $\mathbb{R}$ or $\mathbb{C}$ unless otherwise specified.

Proposition 2.1.7: Weak Topology is Locally Convex

Let $\langle \cdot, \cdot \rangle : V \otimes V \to k$ be a dual pairing of a vector space with itself. Then the weak topology on V gives V the structure of a locally convex topological vector space.

Definition 2.1.8: Graded Frobenius Algebra

A **graded Frobenius $*$-algebra** $(F, \langle \cdot, \cdot \rangle)$, is a graded not-necessarily associative nor unital algebra F over a $*$-ring k, with non-degenerate bilinear form $\langle \cdot, \cdot \rangle : F \otimes F \to k$, such that

$$\langle xy, z \rangle = \langle x, yz \rangle \tag{2.2}$$

for all $x, y, z \in F$ and for all non-zero $x \in F$, there exist some $a, b \in F$ such that ax and $xb \neq 0$.

Our notion of Frobenius algebra generalizes the standard notion in four ways, permitting: non-associative algebras, infinite-dimensional algebras, a graded structure, and an involution.

$\boxed{\text{Trivial Structure:}}$ In the case of a finite-dimensional algebra concentrated in degree zero, with trivial involution, we recover the usual notion of Frobenius algebra.

$\boxed{\text{Non-Associativity:}}$ The condition on the pairing, $\langle xy, z\rangle = \langle x, yz\rangle$ can be thought of as a kind of weak associativity condition. We will see that it is equivalent to the existence of a linear functional ε on the algebra such that $\varepsilon((xy)z) = \varepsilon(x(yz))$ for all $x, y, z \in F$.

$\boxed{\text{Infinite Dimensions:}}$ In the literature, infinite-dimensional Frobenius algebras are often defined differently to the definition we take here, requiring a map $\Delta : F \to F \otimes F$, called a **comultiplication**, which has to satisfy certain compatibility conditions with the multiplication on F. For more information, see [4, 1].

$\boxed{\text{Graded Structure:}}$ From a physics perspective, allowing graded Frobenius algebras will give us the flexibility to describe both bosons (commuting elements) and fermions (anticommuting elements), by placing the fermionic elements in degree 1. This is a common approach in studying supersymmetry.

$\boxed{\text{Involution:}}$ Defining Frobenius algebras and involutions in the way we have will allow us to treat real and sesquilinear pairings as two sides of the same structure. We make this point more explicitly in Lemma 2.1.9 below.

$\boxed{\text{Integrals:}}$ In what follows, we will think of the Frobenius form as a kind of integral: $\langle f, g\rangle \sim \int fg$.

$\boxed{\text{Non-unital:}}$ Since we would like to think of Frobenius algebras as something like algebras of integrable functions, we don't expect them to arise as unital algebras naturally. The weaker condition that there are no non-zero $x \in F$ which behave like zero elements will be sufficient for our purposes.

Lemma 2.1.9

Let F be a graded Frobenius $*$-algebra with bilinear form $\langle\cdot,\cdot\rangle_{\mathrm{Frob}}$. Then there is an associated sesquilinear form $\langle\cdot,\cdot\rangle$, defined by

$$\langle x, y\rangle \equiv \langle x, y^*\rangle_{\mathrm{Frob}} \tag{2.3}$$

which satisfies

$$\langle xy, z\rangle = \langle x, y^* z\rangle\,. \tag{2.4}$$

Similarly, given a sesequilinear form on F, $\langle\cdot,\cdot\rangle$, satisfying Equation (2.4), there is an associated bilinear form $\langle\cdot,\cdot\rangle_{\mathrm{Frob}}$ that gives F the structure of a Frobenius algebra, defined by

$$\langle x, y\rangle_{\mathrm{Frob}} \equiv \langle x, y^*\rangle\,. \tag{2.5}$$

Proof. $\boxed{\text{If Bilinear:}}$ Given a Frobenius form $\langle \cdot, \cdot \rangle_{\text{Frob}}$, we have

$$\begin{aligned}
\langle xy, z \rangle &= \langle xy, z^* \rangle_{\text{Frob}} \\
&= \langle x, yz^* \rangle_{\text{Frob}} \\
&= \langle x, (y^*)^* z^* \rangle_{\text{Frob}} \\
&= \langle x, (y^* z)^* \rangle_{\text{Frob}} \\
&= \langle x, y^* z \rangle .
\end{aligned}$$

The induced pairing is sesquilinear, as

$$\begin{aligned}
\langle \alpha x + \beta y, \gamma z + \delta w \rangle &= \langle \alpha x + \beta y, (\gamma z + \delta w)^* \rangle_{\text{Frob}} \\
&= \langle \alpha x + \beta y, \gamma^* z^* + \delta^* w^* \rangle_{\text{Frob}} \\
&= \alpha \gamma^* \langle x, z^* \rangle_{\text{Frob}} + \alpha \delta^* \langle x, w^* \rangle_{\text{Frob}} + \beta \gamma^* \langle y, z^* \rangle_{\text{Frob}} + \beta \delta^* \langle y, w^* \rangle_{\text{Frob}} \\
&= \alpha \gamma^* \langle x, z \rangle + \alpha \delta^* \langle x, w \rangle + \beta \gamma^* \langle y, z \rangle + \beta \delta^* \langle y, w \rangle .
\end{aligned}$$

$\boxed{\text{If sesquilinear:}}$ A symmetric argument produces a Frobenius form from a sesquilinear one satisfying Equation (2.4). □

It follows that the structure of a Frobenius algebra can be expressed in terms of either a bilinear or sesquilinear form, satisfying the appropriate associativity conditions.

Definition 2.1.10: Topology on a Frobenius Algebra

Let F be a graded Frobenius algebra over a field k. We view F as a locally convex topological vector space with the weak topology induced by the form

$$\langle \cdot, \cdot \rangle_{\text{Frob}} : F \times F \to k.$$

Note: It is not necessarily the case that the involution on a Frobenius $*$-algebra will be continuous in the above topology. Hence it is not necessarily the case that $\langle x, - \rangle \equiv \langle x, (-)^* \rangle_{\text{Frob}}$ will be a continuous functional for any fixed x.

Lemma 2.1.11: Frobenius Algebra Topology is Induced from Vector Space Basis

Let F be a vector space with bilinear form $\langle -, - \rangle$, and $B = \{b_i\}$ be any basis and let

$$X_F = \{\langle -, x \rangle, \langle x, - \rangle \mid x \in F\}$$
$$X_B = \{\langle -, b_i \rangle, \langle b_i, - \rangle \mid b_i \in B\}.$$

If τ_F is the initial topology on F induced by the maps in X_F and τ_B is the initial topology induced by X_B, then

$$\tau_B = \tau_F.$$

Proof. Since $X_B \subseteq X_F$, $\tau_B \subseteq \tau_F$. For any finite choice of elements $v_1, \ldots, v_t$, and $r > 0$, let

$$U_r(v_1, \ldots, v_t) \equiv \{x \in F \mid |\langle x, v_i \rangle|, |\langle v_i, x \rangle| < r \text{ for all } i\}.$$

Then $\{U_r(v_1, \ldots, v_t) \mid r > 0, t \in \mathbb{N}, v_i \in F\}$ forms a neighbourhood basis of zero in τ_F.

$\boxed{\tau_F \subseteq \tau_B\text{:}}$ For any $U_r(v_1, \ldots, v_t)$ we can express each v_i in the basis B. In particular, we will only need finitely many basis elements to do so, say $\{b_{j_1}, \ldots b_{j_m}\}$. Write $v_i = \sum_\ell \alpha_{i\ell} b_{j_\ell}$. Now consider the open set

$$V = \{x \mid |\langle x, b_{j_\ell} \rangle|, |\langle b_{j_\ell}, x \rangle| < \frac{r}{m(\max_{i\ell} |\alpha_{i\ell}|)} \text{ for all } j_\ell\}.$$

Then for any $x \in V$ we have

$$\begin{aligned} |\langle x, v_i \rangle| &= |\sum_\ell \alpha_{i\ell} \langle x, b_{j_\ell} \rangle| \\ &\leq \sum_\ell |\alpha_{i\ell}| |\langle x, b_{j_\ell} \rangle| \\ &< \sum_\ell \frac{r|\alpha_{i\ell}|}{m(\max_{i\ell} |\alpha_{i\ell}|)} \\ &\leq r. \end{aligned}$$

Similarly, $|\langle v_i, x \rangle| < r$ for all i. Thus $V \subseteq U_r(v_1, \ldots, v_t)$. It follows that $\tau_F \subseteq \tau_B$. □

Lemma 2.1.12

Let V, W be topological vector spaces. Then a bilinear map $\langle -, - \rangle : V_1 \times V_2 \to W$ is continuous if and only if it is continuous at $(0, 0)$.

Proof. Say $\langle -,- \rangle$ is continuous at $(0,0)$. Now take an arbitrary point $(x,y) \in V_1 \times V_2$, and an open neighbourhood U of $\langle x,y \rangle \in W$. Since scalar multiplication and translation are homeomorphisms, $U - \langle x,y \rangle$ is an open neighbourhood of $0 \in W$, and there is a balanced open neighbourhood of 0, $B \subseteq W$ such that

$$B + B + B \subseteq U - \langle x,y \rangle .$$

Now since $\langle -,- \rangle$ is continuous at $(0,0)$ and $\langle 0,0 \rangle = 0$ by bilinearity, we can find a basic open set $A_1 \times A_2 \subseteq V_1 \times V_2$ such that

$$\langle -,- \rangle (A_1 \times A_2) \subseteq B.$$

Without loss of generality, since A_1, A_2 are open sets containing zero, we can take them to be balanced and absorbing. Hence there are scalars

$$0 < \lambda, \mu < 1$$

such that $\lambda x \in A_1$ and $\mu y \in A_2$. Consider the open neighbourhood of (x,y)

$$\mu A_1 \times \lambda A_2 + \langle x,y \rangle .$$

Given $(a,b) \in A_1 \times A_2$, we have

$$\begin{aligned} \langle \mu a + x, \lambda b + y \rangle &= \mu\lambda \langle a,b \rangle + \lambda \langle x,b \rangle + \mu \langle a,y \rangle + \langle x,y \rangle \\ &= \mu\lambda \langle a,b \rangle + \langle \lambda x,b \rangle + \langle a,\mu y \rangle + \langle x,y \rangle . \end{aligned}$$

Since $0 < \lambda, \mu < 1$ and B is balanced, we have $\mu\lambda \langle a,b \rangle \subseteq \mu\lambda B \subseteq B$.

We also have $\langle \lambda x, b\rangle, \langle a, \mu y\rangle \in B$. So

$$\begin{aligned}\langle -, -\rangle (\mu A_1 \times \lambda A_2 + \langle x, y\rangle) &\subseteq B + B + B + \langle x, y\rangle \\ &\subseteq (U - \langle x, y\rangle) + \langle x, y\rangle \\ &= U.\end{aligned}$$

Thus $\langle -, -\rangle$ is continuous at (x, y), and hence everywhere. The other direction of the proof is trivial. $\square$

Lemma 2.1.13: Finite-Dimensional Frobenius Algebra Pairings are Jointly Continuous

Let F be a finite-dimensional Frobenius algebra, viewed as a topological vector space with topology induced from $\langle x, -\rangle_{\text{Frob}}, \langle -, y\rangle_{\text{Frob}}$. Then the pairing

$$\langle \cdot, \cdot\rangle_{\text{Frob}} : F \times F \to k$$

is jointly continuous. Similarly, if F is over a trivially-starred field, and we give F the topology induced by $\langle x, -\rangle, \langle -, y\rangle$, then

$$\langle \cdot, \cdot\rangle : F \times F \to k$$

is jointly continuous.

Proof. We prove that $\langle \cdot, \cdot\rangle_{\text{Frob}}$ is jointly continuous in the appropriate topology. The proof for $\langle \cdot, \cdot\rangle$ is analogous. Let $\{b_i\}$ be a basis of F and consider the following preimages of open balls around zero in k:

$$B_R^i(r) \equiv \langle -, b_i \rangle_{\text{Frob}}^{-1} B_0(r)$$
$$B_L^i(r) \equiv \langle b_i, - \rangle_{\text{Frob}}^{-1} B_0(r).$$

$\boxed{\text{Basis:}}$ From Lemma 2.1.11 a neighbourhood basis of zero in F is given by $\{\bigcap_{i=1}^n B_L^i(r) \cap B_R^i(r) \mid \text{for all } r > 0\}$.

$\boxed{\text{Joint continuity:}}$ From Lemma 2.1.12, it's enough to prove continuity at zero. Consider a ball of radius r around zero $B_0(r)$ in k. We take the open neighbourhood of $(0,0)$

$$V = (\bigcap_{i=1}^n B_L^i(r) \cap B_R^i(r)) \times (\bigcap_{i=1}^n B_L^i(r) \cap B_R^i(r)).$$

Then $\langle \cdot, \cdot \rangle_{\text{Frob}} (V) \subseteq B_0(r)$. Thus the pairing $\langle \cdot, \cdot \rangle_{\text{Frob}}$ is continuous at $(0,0)$ and hence is jointly continuous. □

$\boxed{\text{Convention:}}$ In what follows, $\langle \cdot, \cdot \rangle$ will denote the sesquilinear form unless otherwise specified.

2.1.2 Coalgebras

The standard notion of Frobenius algebra is often expressed in terms of a compatible algebra/coalgebra structure on a vector space. However, as we will see, this formulation is unique to finite-dimensional spaces.

Definition 2.1.14: Coalgebra

A **coalgebra** is a k-vector space C, along with k-linear maps $\varepsilon : C \to k$ and $\Delta : C \to C \otimes C$, such that

$\boxed{\text{Counit Condition:}}$ $(\varepsilon \otimes 1) \circ \Delta(x) = (1 \otimes \varepsilon) \circ \Delta(x) = x$ for all $x \in C$.

$\boxed{\text{Coassociativity Condition:}}$ $(\Delta \otimes 1) \circ \Delta(x) = (1 \otimes \Delta) \circ \Delta(x)$ for all $x \in C$.

$\boxed{\text{Terminology:}}$ We call Δ the **comultiplication** and ε the **counit** of the coalgebra.

$\boxed{\text{Sweedler Notation:}}$ In what follows, we will often employ *Sweedler notation*, denoting the elements $\Delta(x) = \sum_i x_{i,1} \otimes x_{i,2}$ by

$$\sum x_{(1)} \otimes x_{(2)}. \tag{2.6}$$

$\boxed{\text{Notation:}}$ Given a linear map $\phi : C \to C$, we will often denote the linear map $\phi \otimes 1 : C \otimes C \to C \otimes C$, by ϕ_1, and $1 \otimes \phi : C \otimes C \to C \otimes C$, by ϕ_2.

Coalgebras often appear as spaces which become dualized. Linear maps out of a coalgebra, into an algebra, always have the structure of an algebra, called a **convolution algebra**. For proofs of the following three results, see the references below.

Lemma 2.1.15: Convolution Algebra

Given any coalgebra C and algebra A, the set of linear maps $\mathrm{Mor}(C, A)$ has an algebra structure under the **convolution product**:

$$\phi * \psi(x) \equiv \mu(\phi \otimes \psi)(\Delta x) \tag{2.7}$$

where $\mu(a, b) = ab$ is the multiplication in A.

Proof. See [32, Chapter VI Lemma 2.1]. □

Proposition 2.1.16

Let F be a finite-dimensional associative algebra over a field, with multiplication μ. Then F can be given the structure of a Frobenius algebra if and only if there exist $\Delta : F \to F \otimes F$ and $\varepsilon : F \to k$ that make F a coalgebra, and

$$(\mu \otimes 1)(1 \otimes \Delta) = (1 \otimes \mu)(\Delta \otimes 1) = \Delta\mu. \tag{2.8}$$

Proof. See [20, Proposition 2.3.22 and Proposition 2.3.24]. □

Lemma 2.1.17: •

If F is both a unital algebra and counital coalgebra, such that the multiplication and comultiplication μ, Δ satisfy the compatibility condition

$$(\mu \otimes 1)(1 \otimes \Delta) = (1 \otimes \mu)(\Delta \otimes 1) = \Delta\mu, \tag{2.9}$$

then F is finite-dimensional.

Proof. See [20, Proposition 2.3.24]. □

2.2 Properties of Frobenius Algebras

Below we catalogue some basic properties of Frobenius algebras which will be useful later in the work. These results are standard in the case of ordinary (non-graded, non-star, finite-dimensional, associative) Frobenius algebras, and can be found in [20, Chapter 2].

Lemma 2.2.1

Let F be an **associative** unital graded Frobenius algebra. Then there is a linear functional $\varepsilon : F \to k$ whose kernel contains no non-trivial ideals (left nor right), such that

$$\langle x, y \rangle = \varepsilon(xy^*). \tag{2.10}$$

Conversely, given any non-zero functional ε with simple kernel on an associative algebra, $\langle x, y \rangle = \varepsilon(xy^*)$ defines a (sesquilinear) Frobenius form.

Proof. $\boxed{\text{Given a form:}}$ Since F is unital, we can define a functional $\varepsilon(y) \equiv \langle 1, y^* \rangle$. Then $\varepsilon(xy^*) = \langle 1, (xy^*)^* \rangle = \langle 1, x^*y \rangle = \langle x, y \rangle$. Now let I be a right ideal in the kernel of ε and let $x \in I$. It follows that $xy^* \in I$ for all y, so $\langle x, y \rangle = \varepsilon(xy^*) = 0$ for all y. Thus x must be zero, and I trivial. Similarly, if I is a left ideal, taking $y \in I$ would imply $\langle x, y^* \rangle = \varepsilon(xy) = 0$ for all x, and so I must be trivial.

Given a functional: $\langle xy, z\rangle = \varepsilon(xyz^*) = \varepsilon(x(y^*z)^*) = \langle x, y^*z\rangle$. Now if $\langle x, y\rangle = 0$ for all x, we can take the left ideal generated by y^*, $I^L_{y^*}$, which will be contained in the kernel of ε. Since ε is non-zero by assumption, it follows that $I^L_{y^*}$ must be trivial, so $y^* = 0$. Using the linearity of the involution, we conclude $y = y^{**} = 0$. Similarly, if $\langle x, y\rangle = 0$ for all y, the right ideal generated by x must be trivial, and thus x must be zero. It follows that the pairing is non-degenerate. $\square$

Terminology: We call the linear functional ε, the **counit** of the Frobenius algebra.

Lemma 2.2.2: Tensor Products of Frobenius Algebras

Let A, B be graded (not-necessarily associative) Frobenius algebras. Then their tensor product as graded algebras $A \otimes B$ has a Frobenius algebra structure, with the pairing

$$\langle x \otimes y, a \otimes b\rangle \equiv (-1)^{|a||y|} \langle x, a\rangle_A \langle y, b\rangle_B \tag{2.11}$$

which we extend bilinearly.

Proof. $\boxed{\text{Associativity Condition:}}$ First note that we can rewrite

$$\begin{aligned}
&\langle (x \otimes y)(u \otimes v), a \otimes b \rangle \\
&= (-1)^{|y||u|} \langle (xu \otimes yv), a \otimes b \rangle \\
&= (-1)^{|y||u|}(-1)^{|yv||a|} \langle xu, a \rangle \langle yv, b \rangle \\
&= (-1)^{|y||u|+|yv||a|} \langle x, u^*a \rangle \langle y, v^*b \rangle \\
&= (-1)^{|y||u|+|yv||a|}(-1)^{|y||u^*a|} \langle x \otimes y, u^*a \otimes v^*b \rangle \\
&= (-1)^{|y||u|+|yv||a|+|y||u^*a|}(-1)^{|v^*||a|} \langle x \otimes y, (u \otimes v)^*(a \otimes b) \rangle .
\end{aligned}$$

Now the exponent of (-1) can be rewritten, noting that the star involution is a degree 0 map, as

$$2|y||u| + 2|v||a| + 2|y||a|.$$

It follows that the factor of (-1) disappears, so

$$\langle (x \otimes y)(u \otimes v), a \otimes b \rangle = \langle x \otimes y, (u \otimes v)^*(a \otimes b) \rangle .$$

For more general tensor elements, we then have

$$\begin{aligned}
\left\langle (\sum_{ij} x_i \otimes y_j)(\sum_{k\ell} u_k \otimes v_\ell), \sum_{mn} a_m \otimes b_n \right\rangle &= \sum_{ijklmn} \langle x_i \otimes y_j, (u_k \otimes v_\ell)^*(a_m \otimes b_n) \rangle \\
&= \left\langle \sum_{ij} x_i \otimes y_j, (\sum_{k\ell} u_k \otimes v_\ell)^* \sum_{mn} a_m \otimes b_n \right\rangle .
\end{aligned}$$

$\boxed{\text{Non-degeneracy:}}$ The non-degeneracy of a pairing $\langle \cdot, \cdot \rangle : V \otimes V \to k$ is equivalent to injections into the dual spaces $V \to V^*$ of the form

$v \to \langle v, - \rangle$ and $v \to \langle -, v \rangle$. Consider the map $A \otimes B \to A^* \otimes B^*$ given by

$$a \otimes b \to \langle a, - \rangle \otimes \langle b, - \rangle \,. \tag{2.12}$$

This is the tensor product of injective maps. Since we're working with vector spaces, which are in particular flat modules, we find that this map is also injective. Similarly, the map $a \otimes b \to \langle -, a \rangle \otimes \langle -, b \rangle$ is injective. It follows that composing with the injection $A^* \otimes B^* \to (A \otimes B)^*$, given by the pointwise product, recovers our pairing and is also injective. $\square$

2.3 Examples of Frobenius Algebras

2.3.1 Examples from Algebra and Geometry

Example 2.3.1: Frobenius Algebras: Vector Spaces with Chosen Basis

For any vector space, V, pick a basis $\{e_i\}$ and define the multiplication and counit by

$$\begin{aligned} e_i e_j &= \delta_{i,j} e_i \\ \varepsilon(e_i) &= 1. \end{aligned} \tag{2.13}$$

If V is finite-dimensional, the comultiplication on this Frobenius algebra is given by

$$\Delta e_i = e_i \otimes e_i. \tag{2.14}$$

$\boxed{\text{Frobenius Form:}}$ If $x = \sum_i x_i e_i$, it follows that

$$\langle x, y \rangle = \varepsilon(xy^*) = \sum_i x_i y_i^*$$

is the ordinary inner product in the case that V is a real or complex vector space.

$\boxed{\text{Associative:}}$ Note that

$$\begin{aligned}(e_i e_j)e_k &= \delta_{ij} e_i e_k = \delta_{ij}\delta_{ik} e_i \\ &= \delta_{ijk} e_i \\ &= \delta_{ij}\delta_{jk} e_i \\ &= e_i(e_j e_k).\end{aligned}$$

It follows that $(xy)z = \sum_{ijk} x_i y_j z_k (e_i e_j) e_k = \sum x_i y_j z_k e_i (e_j e_k) = x(yz)$, so the multiplication is associative.

Definition 2.3.2: Clifford Algebra

Let V be a finite-dimensional vector space over a field k with char $k \neq 2$. Let $Q : V \to k$ be a quadratic form. The **Clifford algebra** associated to (V, Q) is the quotient algebra

$$T(V)/\left\langle x^2 \sim Q(x)1 \right\rangle,$$

where $T(V)$ is the tensor algebra on V.

Below we make use of several results from the theory of Clifford algebras. For more information, see [21, Chapter V].

$\boxed{\text{Pairing:}}$ Every Clifford algebra comes with a symmetric pairing induced by Q:

$$\langle x, y \rangle_Q \equiv Q(x+y) - Q(x) - Q(y).$$

For any field k with $\operatorname{char} k \neq 2$, any symmetric bilinear form admits an orthogonal basis. Writing $Q(x+y)1 = (x+y)^2 = xy + yx + x^2 + y^2 = xy + yx + Q(x)1 + Q(y)1$, we have

$$\begin{aligned} \langle x, y \rangle_Q 1 &= (Q(x+y) - Q(x) - Q(y))1 \\ &= xy + yx. \end{aligned}$$

For orthogonal basis elements $\{e_i\}$, we then have $e_i e_j = -e_j e_i$. We also have $\langle e_i, e_i \rangle_Q = Q(2e_i) - Q(e_i) - Q(e_i) = 4Q(e_i) - Q(e_i) - Q(e_i) = 2Q(e_i)$. So

$$\langle e_i, e_j \rangle = 2\delta_{ij} Q(e_i).$$

Example 2.3.3: Frobenius Algebra: Clifford Algebra with Basis

Let C be a Clifford algebra with non-degenerate quadratic form Q, with associated orthogonal basis $\{e_i\}$. Let E be the subspace of monomials in the $\{e_i\}$:

$$E = \{\sum_{k\geq 1}\sum_{\bar{i}} \alpha_{\bar{i}} e_{i_1} \dots e_{i_k}\}$$

where $\bar{i}$ ranges over the multi-indices $(i_1 < \dots < i_k)$ of length k. Then C is a Frobenius algebra with trivial involution and the counit

$$\begin{aligned} \varepsilon(1) &= 1 \\ \varepsilon(x) &= 0, \quad x \in E. \end{aligned} \tag{2.15}$$

$\boxed{\text{Clifford pairing:}}$ First note that $e_i e_j \in E$ unless $i = j$, in which case, $e_i^2 = Q(e_i)1$. It follows that

$$\langle e_i, e_j \rangle_{\text{Frob}} = \varepsilon(e_i e_j) = \delta_{ij} Q(e_i) = \frac{1}{2} \langle e_i, e_j \rangle_Q .$$

Now if $x, y \in V$, with $x = \sum_i x_i e_i$ and $y = \sum_i y_i e_i$, we have

$$\begin{aligned} \langle x, y \rangle_{\text{Frob}} &= \sum_{ij} x_i y_i \langle e_i, e_j \rangle_{\text{Frob}} = \frac{1}{2} \sum_{ij} x_i y_i \langle e_i, e_j \rangle_Q \\ &= \frac{1}{2} \langle x, y \rangle_Q . \end{aligned}$$

In particular, $\langle x, y \rangle = \sum_i x_i y_i Q(e_i)$.

$\boxed{\text{Non-degenerate:}}$ Consider an arbitrary product of basis elements of V: $e_{i_1} \dots e_{i_k}$, with $i_1 < \dots < i_k$. By definition, $\varepsilon(e_{i_1} \dots e_{i_k}) = 0$. Now $\langle e_{i_1} \dots e_{i_k}, e_{j_1} \dots e_{j_\ell} \rangle_{\mathrm{Frob}} \neq 0$ if and only if $e_{i_1} \dots e_{i_k}$ is a permutation of $e_{j_1} \dots e_{j_\ell}$ (otherwise the product $e_{i_1} \dots e_{i_k} e_{j_1} \dots e_{j_\ell}$ will be a non-trivial product $e_{m_1} \dots e_{m_n}$, and hence the counit will send it to zero). We also have

$$\begin{aligned} \langle e_{i_1} \dots e_{i_k}, e_{i_k} \dots e_{i_1} \rangle_{\mathrm{Frob}} &= \varepsilon(e_{i_1} \dots e_{i_k}, e_{i_k} \dots e_{i_1}) \\ &= Q(e_{i_1}) \dots Q(e_{i_k}). \end{aligned}$$

It follows that given two elements of $x, y \in C$, which we express in the basis $\{e_{i_1} \dots e_{i_k}\}_{i_1 < \dots < i_k}$, we have

$$\begin{aligned} \langle x, y \rangle_{\mathrm{Frob}} &= \sum_{\substack{i_1 < \dots < i_k \\ j_1 < \dots < j_\ell}} x_{\bar{i}} y_{\bar{j}} \langle e_{i_1} \dots e_{i_k}, e_{j_1} \dots e_{j_\ell} \rangle_{\mathrm{Frob}} \\ &= \sum_{i_1 < \dots < i_k} x_{\bar{i}} y_{\bar{i}} Q(e_{i_1}) \dots Q(e_{i_k}) (-1)^{\frac{k(k-1)}{2}}, \end{aligned}$$

where we used the fact that, given a product $e_{i_1} \dots e_{i_k}$, it takes a total of $(k-1) + (k-2) + \dots + 1 = k(k-1)/2$ transpositions to reorder it into $e_{i_k} \dots e_{i_1}$ (first moving e_{i_k} to the left $k-1$ places, then $e_{i_{k-1}}$ to the left $k-2$ places, and so on). In particular, note that the pairing is symmetric:

$$\langle x, y \rangle_{\mathrm{Frob}} = \langle y, x \rangle_{\mathrm{Frob}} .$$

Now given any non-zero $x \in C$, expressed in the above basis, there is some non-zero coefficient $x_{\bar{j}}$ of $e_{j_1} \dots e_{j_\ell}$. Then we have (using the fact that Q being non-degenerate implies $Q(e_i) \neq 0$ for all i)

$$\langle x, e_{j_1} \dots e_{j_\ell} \rangle = x_{\bar{j}} Q(e_{j_1}) \dots Q(e_{j_\ell})(-1)^{\frac{\ell(\ell-1)}{2}} \neq 0.$$

Thus our pairing is non-degenerate, and we have the structure of a Frobenius algebra.

Example 2.3.4: Real Frobenius $*$-Algebra: Complex Numbers

Consider the real vector space $\mathbb{R}^{2n}$, with basis $\{a_i, b_i\}_{i=1}^n$. We impose the following multiplication

$$\begin{aligned} a_i b_j &= \delta_{ij} b_i \\ b_i a_j &= \delta_{ij} b_i \\ a_i a_j &= \delta_{ij} a_i \\ b_i b_j &= -\delta_{ij} a_i, \end{aligned}$$

take the counit to be $\varepsilon(a_i) = 1$, $\varepsilon(b_i) = 1$, and let the star involution be $a_i^* = a_i$, $b_i^* = -b_i$.

For any fixed i, we have

$$(\alpha_1 a_i + \beta_1 b_i)(\alpha_2 a_i + \beta_2 b_i) = (\alpha_1\alpha_2 - \beta_1\beta_2)a_i + (\alpha_1\beta_2 + \beta_1\alpha_2)b_i,$$

recovering the complex numbers. The induced Frobenius pairing will be

$$\langle \alpha_i a_i + \beta_i b_i, (\alpha_j a_j + \beta_j b_j)^* \rangle_{\text{Frob}} = \delta_{ij}(\alpha_i\alpha_j + \beta_i\beta_j - \alpha_i\beta_j + \beta_i\alpha_j).$$

If we view $z_i = \alpha_i a_i + \beta_i b_i$ as an element of $\mathbb{C}^n$, the above pairing is the composition of the ordinary pairing on $\mathbb{C}^n$ with the $\mathbb{R}$-linear map $\pi : \mathbb{C} \to \mathbb{R}$ that sends $\pi(1) = \pi(i) = 1$.

Example 2.3.5: Real Frobenius $*$-Algebra: Orthogonal Form

Take the same $*$-algebra structure on $\mathbb{R}^{2n}$ as in the previous example, but consider the following counit:

$$\varepsilon_O(a_i) = 1$$
$$\varepsilon_O(b_i) = 0.$$

The Frobenius form in this example is the standard orthogonal form on $\mathbb{R}^{2n}$.

From our previous example, we know that

$$(\alpha_1 a_i + \beta_1 b_i)(\alpha_2 a_i + \beta_2 b_i) = (\alpha_1\alpha_2 - \beta_1\beta_2)a_i + (\alpha_1\beta_2 + \beta_1\alpha_2)b_i.$$

It follows that $\varepsilon((\alpha_1 a_i + \beta_1 b_i)(\alpha_2 a_i + \beta_2 b_i)^*) = \alpha_1\alpha_2 + \beta_1\beta_2$. Thus

$$\varepsilon_O(xy^*) = \langle x, y\rangle_O = \sum_i x_i y_i,$$

yielding the standard inner product.

Example 2.3.6: Real Frobenius $*$-Algebra: Symplectic Form

If we take the same $*$-algebra structure on $\mathbb{R}^{2n}$ as in the previous two examples, but select our counit to be

$$\varepsilon_S(a_i) = 0$$
$$\varepsilon_S(b_i) = 1$$

our Frobenius form becomes the standard symplectic form.

The Frobenius form $\langle x, y \rangle = \varepsilon(xy^*)$ satisfies

$$\langle a_i, a_j \rangle = 0 \qquad \langle a_i, b_j \rangle = -\delta_{ij}$$
$$\langle b_i, b_j \rangle = 0 \qquad \langle b_i, a_j \rangle = \delta_{ij},$$

giving the standard symplectic pairing.

$\boxed{\text{Unitary matrices:}}$ The linear maps which simultaneously preserve both of the above Frobenius structures on $\mathbb{R}^{2n}$ can be identified with the unitary matrices $U(n)$. This result is part of the **2-out-of-3 property** (see for instance [3, Section 41]). This suggests that considering multiple Frobenius algebra structures on the same underlying algebra can be useful. We pursue this line of thought in Chapter 4.

Example 2.3.7: Group characters

Let G be a finite group. The following paring is non-degenerate and defines a Frobenius algebra structure on the group algebra kG:

$$\langle g, h \rangle = \delta_{g^{-1},h}. \tag{2.16}$$

$\boxed{\text{Non-degeneracy:}}$ Given a non-zero $u = \sum_i \alpha_i g_i \in kG$, we have $\langle u, g_1^{-1} \rangle = \alpha_1 \neq 0$. Thus the pairing is non-degenerate.

$\boxed{\text{Associativity Condition:}}$ Let $g, h, k \in G$. Then $\langle gh, k \rangle = \delta_{h^{-1}g^{-1},k} = \delta_{g^{-1},hk} = \langle g, hk \rangle$.

$\boxed{\text{Group Characters:}}$ Given a finite group, G, using the isomorphism noted above, we can view any $\phi : G \to k$ as an element of kG. Applying the above Frobenius form, we have

$$\langle \phi, \psi \rangle \equiv \sum_{g \in G} \phi(g)\psi(g^{-1}) \tag{2.17}$$

which is the inner product on group characters (up to a normalization of $\frac{1}{|G|}$).

Example 2.3.8: Graded $*$-Algebra: Composition Algebra

A **composition algebra** is a finite-dimensional vector space V with not-necessarily associative multiplication, a unit, and a non-degenerate symmetric bilinear form $(\cdot, \cdot)$, such that

$$(xy, xy) = (x, x)(y, y). \tag{2.18}$$

For more information on composition algebras see [33, Chapter 1]

$\boxed{\text{Composition involution:}}$ Every composition algebra has an involution defined by

$$x^\dagger \equiv 2(x, 1)1 - x. \tag{2.19}$$

It turns out that $(xy, z) = (x, zy^\dagger)$ and $(xy)^\dagger = y^\dagger x^\dagger$. We can define a new non-degenerate pairing on any composition algebra by

$$\langle x, y\rangle_{\text{Frob}} \equiv (x, y^\dagger), \tag{2.20}$$

which will satisfy $\langle xy, z\rangle_{\text{Frob}} = \langle x, yz\rangle_{\text{Frob}}$.

$\boxed{\text{Frobenius Involution:}}$ Note that the involution on the composition algebra is anti-multiplicative: $(xy)^\dagger = y^\dagger x^\dagger$, but the involution on our Frobenius algebras are defined to be *multiplicative*. So we view our underlying ring k as a $*$-ring with trivial involution and take our involution on the Frobenius algebra to be trivial as well.

$\boxed{\text{Complex Numbers:}}$ The complex numbers can be viewed as a real composition algebra, where

$$(x, y) \equiv \frac{1}{2}(x^* y + x y^*).$$

The involution on this algebra is the complex conjugate: $x^\dagger = x^*$, and the Frobenius form we end up with is the real part of the product:

$$\langle x, y\rangle_{\text{Frob}} = \mathbb{R}(xy).$$

On the other hand, taking the complex numbers as a complex composition algebra, where

$$(x, y) = xy$$

and the trivial involution is $x^\dagger = x$ gives

$$\langle x, y\rangle_{\text{Frob}} = xy.$$

Note in this case, even though the composition algebra involution $(-)^\dagger$ is trivial, on our complex Frobenius algebra we can take the complex conjugate as our involution, giving

$$\langle x, y \rangle = xy^*,$$

the ordinary complex inner product. Thus the involutions in the composition and Frobenius algebras play different roles in general.

$\boxed{\text{Octonions:}}$ The octonions can be viewed as a real composition algebra as follows: let $\{e_0, \ldots, e_7\}$ be the standard basis of the octonions, and write each $x = \sum_i x_i e_i$. Then

$$(x, y) = \sum_i x_i y_i$$

and $x^\dagger = x_0 e_0 - \sum_{i=1}^7 x_i e_i$. We get

$$\langle x, y \rangle_{\text{Frob}} = x_0 y_0 - \sum_{i=1}^7 x_i y_i.$$

This provides a nice example of a Frobenius algebra which is not associative, but still satisfies associativity under the pairing:

$$\langle xy, z \rangle = \langle x, yz \rangle .$$

We can think of Frobenius algebras of this type as being *weakly associative.*

Example 2.3.9: Frobenius Algebra: Matrix Algebras

Let $M_n(\mathbb{R})$ be the algebra of $n \times n$ matrices. This can be given a Frobenius algebra structure, with pairing

$$\langle A, B \rangle_{\text{Frob}} = \operatorname{Tr}(AB). \tag{2.21}$$

$\boxed{\text{Complex case:}}$ If we instead take $M_n(\mathbb{C})$ $n \times n$ complex matrices, viewed as either a real or complex algebra, we can define an involution $* : A \to A$ by entrywise complex conjugation, which gives A the structure of a $*$-Frobenius algebra.

2.3.2 Examples from Functional Analysis

Example 2.3.10: Frobenius Algebra: Hilbert Space with Basis

Let H be a Hilbert space (even one that is non-separable). Pick an orthonormal basis $\{e_\alpha\}$ and define a multiplication as follows: given any two elements $x = \sum_{i=1}^{\infty} x_{\alpha_i} e_{\alpha_i}$, $y = \sum_{j=1}^{\infty} y_{\beta_j} e_{\beta_j}$, let

$$xy \equiv \sum_{k=1}^{\infty} x_{\gamma_k} y_{\gamma_k} e_{\gamma_k},$$

where $\{e_{\gamma_k}\}_{k=1}^{\infty} = \{e_{\alpha_i}\}_{i=1}^{\infty} \cup \{e_{\beta_j}\}_{j=1}^{\infty}$, and the coefficients x_i, y_j have been reindexed appropriately. With the pairing

$$\langle x, y\rangle_{\text{Frob}} \equiv \sum_{k=1}^{\infty} x_{\gamma_k} y_{\gamma_k},$$

this has the structure of a Frobenius algebra. For more information on non-separable Hilbert spaces, see [2].

We need to check that the multiplication is well-defined: that it converges and doesn't depend on how we chose to order the e_{γ_k} in the sum.

$\boxed{\text{Pythagorean Theorem:}}$ Since $\{e_\alpha\}$ are orthonormal, we have

$$\begin{aligned}\left\|\sum_{i=1}^{N}\lambda_i e_{\alpha_i}\right\|^2 &= \left\langle \sum_{i=1}^{N}\lambda_i e_{\alpha_i}, \sum_{i=1}^{N}\lambda_i e_{\alpha_i}\right\rangle_H = \sum_{i,j=1}^{N}\left\langle \lambda_i e_{\alpha_i}, \lambda_j e_{\alpha_j}\right\rangle_H \\ &= \sum_{i=1}^{N}|\lambda_i|^2.\end{aligned}$$

$\boxed{\text{Convergence:}}$ First note if $x = \sum_i x_i e_{\alpha_i}$, then $(\sum_{i=1}^N x_i e_{\alpha_i})_{N=1}^\infty$ is a Cauchy sequence. Considering the differences of the partial sums, we have

$$\begin{aligned}\left\|\sum_{i=1}^{N}x_i e_{\alpha_i} - \sum_{i=1}^{M}x_i e_{\alpha_i}\right\| &= \left\|\sum_{i=M+1}^{N}x_i e_{\alpha_i}\right\| \\ &= \sum_{i=M+1}^{N}|x_i|^2.\end{aligned}$$

It follows that the sequence $(\sum_{i=1}^N x_i e_{\alpha_i})$ is Cauchy if and only if $(\sum_{i=1}^N |x_i|^2)$ is Cauchy. Now if x and y are expressed in terms of our orthonormal basis, we can show that $\sum_k x_{\gamma_k} y_{\gamma_k} e_{\gamma_k}$ is absolutely convergent: for any fixed M, N, by the Cauchy-Schwarz inequality,

$$\sum_{k=M}^{N}|x_{\gamma_k}y_{\gamma_k}| \le \sqrt{\sum_{k=M}^{N}|x_{\gamma_k}|^2\sum_{\ell=M}^{N}|y_{\gamma_\ell}|^2}.$$

Since $(\sum_{k=1}^N |x_{\gamma_k}|^2)$ and $(\sum_{\ell=1}^N |y_{\gamma_\ell}|^2)$ are Cauchy by the argument above, we can make the right hand side of the inequality arbitrarily small by picking M, N sufficiently large. Thus the series $\sum_k |x_{\gamma_k} y_{\gamma_k}|$ is absolutely convergent. In a Hilbert space, absolute convergence implies unconditional convergence. Thus the series

$$\sum_{k=1}^{\infty} x_{\gamma_k} y_{\gamma_k} e_{\gamma_k}$$

is both convergent and independent of how the terms in the sum are rearranged. So our multiplication is well-defined.

$\boxed{\text{Non-degeneracy:}}$ First note that our pairing

$$\langle x, y \rangle_{\text{Frob}} \equiv \sum_{k=1}^{\infty} x_{\gamma_k} y_{\gamma_k}$$

is well-defined, since the sum on the right is absolutely convergent from the argument above, and thus converges. Note that the Frobenius pairing matches the inner product on the Hilbert space. Now given any non-zero x, there is some coefficient $x_{\alpha_j} \neq 0$. Then

$$\left\langle x, e_{\alpha_j} \right\rangle = x_{\alpha_j} \neq 0,$$

and so our pairing is non-degenerate.

$\boxed{\text{Associativity:}}$

$$\begin{aligned} \langle x, yz \rangle_{\text{Frob}} &= \left\langle x, \sum_k y_{\gamma_k} z_{\gamma_k} e_{\gamma_k} \right\rangle = \sum_{\ell} x_{\theta_\ell} y_{\theta_\ell} z_{\theta_\ell} \\ &= \langle xy, z \rangle_{\text{Frob}} . \end{aligned}$$

Both pairings are equal to the sum of the product of those coefficients of x, y, z that are non-zero on the same basis elements.

Example 2.3.11: Frobenius Algebra: Compactly Supported Functions

Let $C_c^\infty(\mathbb{R}^n)$ be the algebra of smooth compactly supported functions. The pairing

$$\langle f, g \rangle_{\text{Frob}} = \int fg \tag{2.22}$$

gives $C_c^\infty(\mathbb{R}^n)$ the structure of a Frobenius algebra.

This example works more generally: Let M be any oriented smooth manifold equipped with a volume form $\omega \in \Omega^n(M)$. The chosen volume form gives us a way of defining an integral over the manifold. Taking the algebra $C_c^\infty(M)$ of smooth compactly supported functions, we can define a Frobenius form by

$$\langle f, g \rangle_{\text{Frob}} = \int_M fg\omega,$$

giving $C_c^\infty(M)$ the structure of a Frobenius algebra.

Example 2.3.12: Frobenius Algebra: Schwartz Space

Let

$$\mathcal{S} = \{f \in C^\infty(\mathbb{R}^n, \mathbb{C}) \mid \forall \alpha, \beta, \sup_{x \in \mathbb{R}^n} |x_1^{\alpha_1} \dots x_n^{\alpha_n} (\partial_1^{\beta_1} \dots \partial_n^{\beta_n} f)| < \infty\}$$

be the algebra of Schwartz functions. The pairing

$$\langle f, g \rangle = \int fg^* \tag{2.23}$$

gives $\mathcal{S}$ the structure of a Frobenius $*$-algebra.

2.3.3 Graded Examples

Example 2.3.13: Graded Frobenius Algebra: Fermionic Integral

Let F be the exterior algebra $\Lambda[\theta_1, \ldots, \theta_n]$. The Berezin integral is a linear functional defined by

$$\begin{aligned} &\varepsilon(\theta_1 \wedge \ldots \wedge \theta_n) = 1 \\ &\varepsilon(x) = 0, \text{ if } x \text{ is missing one of the } \theta_i. \end{aligned} \tag{2.24}$$

Taking the Berezin integral as our counit gives the exterior algebra the structure of a graded Frobenius algebra. For more information on Berezin integrals see [29, Section 9.5].

$\boxed{\text{Grading:}}$ $\Lambda[\theta_1, \ldots, \theta_n]$ is viewed as a graded algebra where products of k many θ_i have degree k.

Example 2.3.14: Graded Frobenius Algebra: Functions of Bosonic and Fermionic Variables

Let $K \subseteq \mathbb{R}^m$ be a compact subset and let F be the graded vector space $S[x_1, \ldots, x_m] \otimes \Lambda[\theta_1, \ldots, \theta_m]$. For any $f = \sum f_{(1)} \otimes f_{(2)}$ we define

$$\varepsilon(f) \equiv \int_K f_{(1)} \varepsilon_B(f_{(2)}) = \varepsilon_B(f_{(2)}) \int_K f_{(1)} \tag{2.25}$$

where ε_B is the Berezin integral. This gives F the structure of a Frobenius algebra.

We can similarly replace $S[x_1, \dots, x_n]$ by any algebra of integrable functions, such as $C_c^\infty(\mathbb{R}^m)$.

Note: The structure of this example is the tensor product of the Frobenius algebra structure of $S[x_1, \dots, x_m]$ given by integrating over K, and the Frobenius algebra of $\Lambda[\theta_1, \dots, \theta_n]$ given by integration with respect to the Berezin integral.

Example 2.3.15: Graded Frobenius Algebra: Super Vector Space

Let $V = V_0 \oplus V_1$ be a graded free module over $\mathbb{R}$ or $\mathbb{C}$. Take fixed bases $\{e_i^0\}$, $\{e_i^1\}$, for V_0 and V_1 respectively. We turn V into a graded algebra with the following multiplication:

$$e_i^k e_j^\ell \equiv (-1)^{|k||\ell|} \delta_{ij} e_i^{k+\ell \text{ (mod 2)}}. \tag{2.26}$$

Choosing the counit to be $\varepsilon(e_0^k) = 1$, $\varepsilon(e_1^k) = \dfrac{1}{2}\hbar$ for some real $\hbar$, gives V the structure of a graded Frobenius algebra.

Associativity: To show associativity, first we consider products of basis elements: $e_{i_1}^{j_1}, e_{i_2}^{j_2}, e_{i_3}^{j_3}$. Note that this product will be zero, no matter how parentheses are placed amongst the three terms, unless $i_1 = i_2 = i_3$. So without loss of generality, we can consider $i_1 = i_2 = i_3 \equiv i$. Now we have

$$
\begin{aligned}
(e_i^j e_i^k)e_i^\ell &= (-1)^{|j||k|} e_i^{j+k \pmod 2} e_i^\ell \\
&= (-1)^{|j||k|}(-1)^{(|j|+|k|)|\ell|} e_i^{j+k+\ell \pmod 2}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
e_i^j(e_i^k e_i^\ell) &= (-1)^{|k||\ell|} e_i^j e_i^{k+\ell \pmod 2} \\
&= (-1)^{|j|(|k|+|\ell|)+|k||\ell|} e_i^{j+k+\ell \pmod 2}.
\end{aligned}
$$

Thus $(e_i^j e_i^k)e_i^\ell = e_i^j(e_i^k e_i^\ell) = (-1)^{|j||k|+|j||\ell|+|k||\ell|} e_i^{j+k+\ell \pmod 2}$. Now an arbitrary product will have the form

$$
\begin{aligned}
(xy)z &= \sum_{\bar{i},\bar{j}} x_{i_1}^{j_1} y_{i_2}^{j_2} z_{i_3}^{j_3} (e_{i_1}^{j_1} e_{i_2}^{j_2}) e_{i_3}^{j_3} \\
&= \sum_{\bar{i},\bar{j}} x_{i_1}^{j_1} y_{i_2}^{j_2} z_{i_3}^{j_3} e_{i_1}^{j_1} (e_{i_2}^{j_2} e_{i_3}^{j_3}) \\
&= x(yz).
\end{aligned}
$$

So our algebra is associative.

$\boxed{\text{Multiplication:}}$ More explicitly, our multiplication will have the form

$$
\begin{aligned}
xy &= \sum_{\bar{i},\bar{j}} x_{i_1}^{j_1} y_{i_2}^{j_2} e_{i_1}^{j_1} e_{i_2}^{j_2} \\
&= \sum_{i,j_1,j_2} (-1)^{|j_1||j_2|} x_i^{j_1} y_i^{j_2} e_i^{j_1+j_2 \pmod 2} \\
&= \sum_i (x_i^0 y_i^0 - x_i^1 y_i^1) e_i^0 + (x_i^0 y_i^1 + x_i^1 y_i^0) e_i^1.
\end{aligned}
$$

In particular,

$$x^2 = \sum_i ((x_i^0)^2 - (x_i^1)^2)e_i^0 + (x_i^0 x_i^1 + x_i^1 x_i^0)e_i^1.$$

Note that this product is **not graded-commutative** since $e_i^1 e_i^1 = -e_i^0 \neq 0$. However, we always have $xy = yx$. It follows that our pairing $\langle x, y\rangle = \varepsilon(xy) = \langle y, x\rangle$ will be symmetric.

$\boxed{\text{Pairing:}}$ If we set $\varepsilon(e_i^0) = 1$ and $\varepsilon(e_i^1) = \frac{1}{2}\hbar$ for some real number $\hbar$, we can view

$$\langle x, x\rangle = \sum_i ((x_i^0)^2 - (x_i^1)^2) + (x_i^0 x_i^1)\hbar \tag{2.27}$$

as a kind of deformation of the **Minkowski metric**.

$\boxed{\text{Non-degeneracy:}}$ Let $x = \sum_{ij} x_i^j e_i^j$ be a non-zero element of V. We have

$$\left\langle x, e_k^\ell \right\rangle = \sum_j x_k^j (-1)^{|j||\ell|}\varepsilon(e_k^{j+\ell}) = x_k^0 \varepsilon(e_k^\ell) + (-1)^{|\ell|} x_k^1 \varepsilon(e_k^{\ell+1}).$$

In particular, $\langle x, e_k^0\rangle = x_k^0 + x_k^1\hbar$ and $\langle x, e_k^1\rangle = x_k^0\hbar - x_k^1$. Thus

$$\left\langle x, \alpha e_k^0 + \beta e_k^1 \right\rangle = (\alpha + \beta\hbar)x_k^0 + (\alpha\hbar - \beta)x_k^1.$$

Now since $x \neq 0$, there is some $x_k^j \neq 0$. We can always choose α and β above to get a non-zero pairing. If $x_k^0 \neq 0$, take $\beta = \alpha\hbar$, so

$$\left\langle x, \alpha e_k^0 + \alpha\hbar e_k^1 \right\rangle = \alpha(1 + \hbar^2)x_k^0 \neq 0.$$

Similarly, if $x_k^1 \neq 0$, then choosing $\alpha = -\beta\hbar$ gives

$$\left\langle x, -\beta\hbar e_k^0 + \beta e_k^1 \right\rangle = -\beta(1+\hbar^2)x_k^1 \neq 0.$$

Thus our pairing is non-degenerate.

Chapter 3

Hopf Algebras and Laplace Pairings

Hopf algebras (and their slightly weaker notion, bialgebras) will play the role of algebras of observables in our formulation of quantum field theory. On their own, these algebras can account for the structure of operators and their products or compositions. However, it is once we introduce additional structure onto our bialgebras, in the form of *Laplace pairings*, that we will be able to account for measurable features of a QFT (the correlators).

3.1 Generalized Bialgebras

Definition 3.1.1: Generalized Bialgebras

Following [23], we define a **generalized bialgebra** with relations R, as a tuple $(B, \mu, \eta, \Delta, \varepsilon, R)$, where B is a k-vector space, equipped with linear maps

$$\begin{aligned} \mu : B \otimes B &\rightleftarrows B : \Delta \\ \eta : k &\rightleftarrows B : \varepsilon \end{aligned} \tag{3.28}$$

such that (B, μ, η) is a unital k-algebra, (B, Δ, ε) is a counital k-coalgebra, and R is a set of relations of the form

$$\delta \circ \theta = \sum_i (\theta_1^i \otimes \ldots \otimes \theta_m^i) \circ \omega^i \circ (\delta_1^i \otimes \ldots \otimes \delta_n^i), \tag{3.29}$$

where $\delta, \delta^i \in \{\Delta, \varepsilon, id\}$, $\theta, \theta^i \in \{\eta, \mu, id\}$ and $\omega^i : B^{\otimes m+n} \to B^{\otimes m+n}$ are braiding maps.

$\boxed{\text{Idea:}}$ The compatibility relations on a generalized bialgebra tell us how we can rewrite compositions of operations followed by co-operations as co-operations followed by operations.

$\boxed{\text{(co)Unital:}}$ In [23], Loday considers generalized bialgebras that are not necessarily unital or counital. Here all of our bialgebras will be both unital and counital unless otherwise specified.

Example 3.1.2: Generalized Bialgebras: Finite-Dimensional Frobenius Algebras

In Proposition 2.1.16 we saw that *finite-dimensional* Frobenius algebras can be characterized as vector spaces that are both algebras and coalgebras, with the compatibility relations

$$\begin{aligned}\Delta(xy) &= \sum xy_{(1)} \otimes y_{(2)} \\ \Delta(xy) &= \sum x_{(1)} \otimes x_{(2)}y.\end{aligned} \tag{3.30}$$

Example 3.1.3: Generalized Bialgebras: Ordinary Bialgebras and Hopf Algebras

What is usually called a bialgebra, is a generalized bialgebra with the compatibility relations

$$\begin{aligned}\Delta(xy) &= \Delta(x)\Delta(y) \\ \varepsilon(xy) &= \varepsilon(x)\varepsilon(y)\end{aligned} \tag{3.31}$$

for all $x, y \in B$.

Definition 3.1.4: Hopf Algebras

A **Hopf algebra** H is a bialgebra with a map $S : H \to H$ (called an **antipode**) such that

$$\eta \circ \varepsilon(x) = \sum x_{(1)}S(x_{(2)}) = \sum S(x_{(1)})x_{(2)}. \tag{3.32}$$

$\boxed{\text{Involution:}}$ If S is an involution, we call H an **involutive Hopf algebra**. In what follows all of our Hopf algebras will be assumed involutive unless otherwise specified.

We will begin by studying ordinary bialgebras and Hopf algebras. However, we will see that certain ideas from quantum physics motivate the study of more general bialgebras, which can be thought of in some cases as deformations of ordinary bialgebras, or bialgebras encoding particular physical properties like particle spin.

3.2 Examples of Ordinary Hopf and Bialgebras

Our basic examples of ordinary bialgebras will be divided into two broad classes: those generated by grouplike elements, and those generated by primitive elements (definitions which we present below).

Definition 3.2.1: Grouplike Elements and Primitive Elements

Let B be a bialgebra. An element $g \in B$ is called **grouplike** if it has the comultiplication

$$\Delta g = g \otimes g, \quad g \neq 0.$$

An element $p \in B$ is called **primitive** if

$$\Delta p = 1 \otimes p + p \otimes 1.$$

We will make use of the following standard results about grouplike and primitive elements below. For proofs, see the reference below.

Proposition 3.2.2: Properties of Grouplike and Primitive Elements

Let B be a bialgebra.

1. The grouplike elements of B form a monoid. Furthermore, if B is a Hopf algebra with antipode S, the monoid of grouplike elements is a group, where the inverse of g is $S(g)$.

2. The primitive elements of B form a Lie algebra under the commutator bracket.

Proof. See [30, Proposition 5.1.15]. □

The following classes of ordinary Hopf and bialgebras will be of two types: those generated by their grouplike elements and those generated by their primitive elements.

Example 3.2.3: Ordinary Bialgebras: kM

Let M be a monoid. The monoid algebra kM has the structure of a bialgebra with

$$\begin{aligned} \Delta x &= x \otimes x \\ \varepsilon(x) &= 1 \end{aligned} \tag{3.33}$$

for all $x \in M$.

$\boxed{\text{Groups:}}$ In the case that our monoid is a group, G, the group algebra kG has Hopf algebra structure with the above comultiplication and counit, and antipode

$$S(g) = g^{-1}. \tag{3.34}$$

It is with this in mind that Hopf algebras are often thought of as generalizations of groups.

$\boxed{\text{Dimensionality:}}$ Note that, unlike Frobenius algebras, Hopf and bialgebras have a compatibility between multiplication and comultiplication that is well-behaved even for infinite-dimensional groups. Unlike Example 2.3.7, the Hopf algebra kG makes no assumptions on the size of G.

Example 3.2.4: Graded Hopf Algebras: Tensor Algebras

Given a graded vector space V, the tensor algebra $T(V)$ is the free graded algebra over V. It has the structure of a Hopf algebra, with

$$\begin{aligned} \Delta x &= x \otimes 1 + 1 \otimes x \\ \varepsilon(x) &= 0 \\ S(x) &= -x \end{aligned} \tag{3.35}$$

for all $x \in V$. We then extend Δ and ε to the rest of $T(V)$ multiplicatively and S anti-multiplicatively.

$\boxed{\text{Note:}}$ If we take V to sit entirely in degree zero, we recover the usual, ungraded version of the tensor algebra.

Example 3.2.5: Graded Hopf Algebra: Exterior Algebra

Given a vector space V, the exterior algebra $\Lambda V = \bigoplus_k \Lambda^k V$ has the structure of a graded Hopf algebra, where $\Lambda^k V$ has degree k, and

$$\begin{aligned} \Delta x &= x \otimes 1 + 1 \otimes x \\ \varepsilon(x) &= 0 \\ S(x) &= -x. \end{aligned} \tag{3.36}$$

for all $x \in V$. We then extend Δ and ε to the rest of $\Lambda(V)$ multiplicatively and S anti-multiplicatively.

Example 3.2.6: Hopf Algebra: Universal Enveloping Algebra

Let $\mathfrak{g}$ be a Lie algebra. The universal enveloping algebra $U(\mathfrak{g})$ has the structure of a Hopf algebra with

$$\begin{aligned} \Delta x &= x \otimes 1 + 1 \otimes x \\ \varepsilon(x) &= 0 \\ S(x) &= -x. \end{aligned} \tag{3.37}$$

for all $x \in \mathfrak{g}$.

Example 3.2.7: Hopf Algebra: Symmetric Algebra

Let $\mathcal{S}(V)$ be the symmetric algebra on the vector space V. This is a Hopf algebra with

$$\begin{aligned} \Delta x &= x \otimes 1 + 1 \otimes x \\ \varepsilon(x) &= 0 \\ S(x) &= -x. \end{aligned} \tag{3.38}$$

Note: The symmetric algebra is a special case of Example 3.2.6, in which we take the abelian Lie algebra V.

Combinatorics: Hopf algebras often arise in combinatorics in the course of enumeration problems. We shall see that in physics, this comes into play in the form of Wick's theorem in perturbative quantum field theory. To give a taste of the combinatorial flavour of Hopf algebras, we present an example below.

Factoring: Let P be the infinite-dimensional vector space consisting of formal linear combinations of the prime numbers over the field k. Using the fact that prime decompositions are unique, the symmetric algebra $\mathcal{S}(P)$ is isomorphic to $k\mathbb{N}$ as an algebra (where $\mathbb{N}$ denotes the monoid of the natural numbers under multiplication). The comultiplication

$$\Delta(p_1^{\alpha_1} \dots p_n^{\alpha_n}) = \sum_k \binom{\alpha_1}{k_1} \cdots \binom{\alpha_n}{k_n} p_{i_1}^{k_1} \dots p_{i_n}^{k_n} \otimes p_{i_1}^{\alpha_1 - k_1} \dots p_{i_n}^{\alpha_n - k_n} \tag{3.39}$$

is the sum of all the ways of expressing the integer $p_1^{\alpha_1} \dots p_n^{\alpha_n}$ as a product of two integers, including multiplicity. We can perform a similar construction for polynomials.

3.3 Graded Bialgebras

As mentioned in Chapter 2, graded structures allow us to encode certain spin statistics: even and odd degree elements of a graded algebra can be thought of physically as bosonic (commuting) and fermionic (anticommuting) elements. However, more exotic spin statistics and commutation relations are of interest in physics. Most famously, the **canonical commutation relation**

$$[x, p] = i\hbar I \tag{3.40}$$

is neither a bosonic nor a femionic condition, and yet we will find that it can still be put onto the same footing as these spin conditions by considering the appropriate kind of generalized bialgebra.

Definition 3.3.1: q-Binomial Coefficients

Let n, k be integers. The ***q*-binomial coefficient** is defined to be

$$\binom{n}{k}_q = \frac{(1-q^n)(1-q^{n-1})\dots(1-q^{n-k+1})}{(1-q)\dots(1-q^k)} \tag{3.41}$$

when $k \leq n$, and 0 otherwise.

$\boxed{\text{Combinatorics:}}$ The q-binomial coefficients can be interpreted as the solution to the following combinatorial problem: given two elements of an algebra x, y such that $xy = qyx$, what is the coefficient of the term $x^k y^{n-k}$ in $(x+y)^n$ after rearranging every term in the expansion into the form $x^a y^b$? In other words, the following q-*binomial formula* holds:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}. \tag{3.42}$$

In the case that x and y commute, we recover the ordinary binomial theorem. The q-binomial theorem is useful in the quantum setting where we often deal with more subtle commutation relations. In the case $q = (-1)$ we recover fermionic (anticommuting) expansions, $q = \hbar$ appears when dealing with canonical commutation relations, $q = e^{i\phi}$ appears in the study of anyons, and more general q appear in the study of quantum groups.

$\boxed{\text{Generalization:}}$ For what follows we would like a form of the binomial theorem that allows us to expand $(x_1+y_1)^{\alpha_1}\dots(x_n+y_n)^{\alpha_n}$.

Lemma 3.3.2: Multivariable q-Binomial Coefficient

Let $\{x_i, y_j\}$ be a collection of elements of an algebra such that

$$x_i y_j = Q_{ij} y_j x_i$$

for all i, j. Then the coefficient of $x_1^{k_1}\dots x_n^{k_n} y_1^{\alpha_1-k_1}\dots y_n^{\alpha_n-k_n}$ in $\prod_i (x_i+y_i)^{\alpha_i}$ is

$$\binom{\alpha_1,\dots,\alpha_n}{k_1,\dots,k_n}_Q = \binom{\alpha_1}{k_1}_{q_1}\dots\binom{\alpha_n}{k_n}_{q_n} \prod_{i_2=1}^{1} Q_{i_2,2}^{(\alpha_{i_2}-k_{i_2})k_2}\dots\prod_{i_n=1}^{n-1} Q_{i_n,n}^{(\alpha_{i_n}-k_{i_n})k_n}, \tag{3.43}$$

where $q_i \equiv Q_{i,i}$.

Proof. Each term in the expansion of $\prod_i (x_i + y_i)^{\alpha_i}$ can be partitioned into blocks of size α_i . We can rearrange each term in the expansion by rearranging the blocks individually, picking up factors of $\prod_i \binom{\alpha_i}{k_i}_{q_i}$, and then passing each of the k_j x_j to the left, through the y_i with $i < j$, picking up the appropriate factors of $Q_{i,j}^{(\alpha_i - k_i)k_j}$. $\square$

$\boxed{\text{Notation:}}$ If we let $Q_{ij} = 1$ when $j \leq i$ and collect the Q_{ij} into a matrix

$$Q = \begin{pmatrix} 1 & Q_{12} & Q_{13} & \dots & Q_{1n} \\ 1 & 1 & Q_{23} & \dots & Q_{2n} \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix},$$

our expression takes the simpler form

$$\binom{\alpha_1, \dots, \alpha_n}{k_1, \dots, k_n} = \prod_{i,j=1}^{n} \binom{\alpha_i}{k_i}_{q_i} Q_{ij}^{(\alpha_i - k_i)k_j}. \tag{3.44}$$

Definition 3.3.3: Braided Tensor Product

Let A, B be algebras with bases $\{e_i^v\}_{i \in I}, \{e_j^w\}_{j \in J}$ respectively, and let $Q_{ji} \in k$ be coefficients for all $i \in I, j \in J$. The **braided tensor product** $A \otimes_Q B$ with respect to Q_{ij} is defined by

$$(e_i^v \otimes e_j^w)(e_k^v \otimes e_\ell^w) = Q_{jk}(e_i^v e_k^v \otimes e_j^w e_\ell^w). \tag{3.45}$$

$\boxed{\text{Idea:}}$ We can interpret Q_{ij} as a kind of deformation or quantization parameter (when dealing with quantum groups and canonical commutation relations), or as capturing the spin of a basis element (when dealing with fermions, bosons, and anyons).

Generalized Bialgebras: Building bialgebras out of the braided tensor product gives us one way of talking about deformed bialgebras or bialgebras with spin. We will call these ***Q*-bialgebras**.

Associativity:

Lemma 3.3.4

Let A be a (not necessarily associative) algebra with basis $\{e_i\}_{i\in I}$ and structure coefficients given by $e_ie_j = \sum_k \alpha_{ij}^k e_k$. Then A is associative if and only if the structure constants satisfy

$$\sum_m \alpha_{ij}^m \alpha_{mk}^n = \sum_m \alpha_{im}^n \alpha_{jk}^m$$

for all $i, j, k, n \in I$.

Proof. It's enough to check associativity for the product of basis elements. Now

$$\begin{aligned}(e_ie_j)e_k &= \sum_m \alpha_{ij}^m e_m e_k \\ &= \sum_{m,n} \alpha_{ij}^m \alpha_{mk}^n e_n.\end{aligned}$$

On the other hand,

$$\begin{aligned}e_i(e_je_k) &= \sum_m \alpha_{jk}^m e_i e_m \\ &= \sum_{m,n} \alpha_{jk}^m \alpha_{im}^n e_n.\end{aligned}$$

Comparing coefficients, we thus have associativity if and only if

$$\sum_m \alpha_{ij}^m \alpha_{mk}^n = \sum_m \alpha_{jk}^m \alpha_{im}^n.$$

□

Lemma 3.3.5

Let A be an associative algebra with basis $\{e_i\}_{i\in I}$ and structure coefficients given by $e_ie_j = \sum_k \alpha_{ij}^k e_k$. Furthermore, let $Q_{ij} \in k$ be coefficients for all $i, j \in I$. Then the multiplication on $A \otimes_Q A$ is associative if and only if

$$\sum_{mn} Q_{jk}\alpha_{ik}^m \alpha_{j\ell}^n Q_{na}\alpha_{ma}^c \alpha_{nb}^d = \sum_{mn} Q_{jm}\alpha_{im}^c \alpha_{jn}^d Q_{\ell a}\alpha_{ka}^m \alpha_{\ell b}^n.$$

Proof. A basis for $A \otimes_Q A$ (as a vector space) is given by $\{e_i \otimes e_j\}_{i,j\in I}$. By definition, the product of these basis elements takes the form

$$\begin{aligned}(e_i \otimes e_j)(e_k \otimes e_\ell) &= Q_{jk} e_ie_k \otimes e_je_\ell \\ &= \sum_{mn} Q_{jk}\alpha_{ik}^m \alpha_{j\ell}^n (e_m \otimes e_n).\end{aligned}$$

Thus the structure coefficients of $A \otimes_Q A$ are $\beta_{ij,k\ell}^{mn} = Q_{jk}\alpha_{ik}^m \alpha_{j\ell}^n$. From Lemma 3.3.4 above, it follows that this product is associative if and only if

$$\sum_{mn} \beta_{ij,k\ell}^{mn} \beta_{mn,ab}^{cd} = \sum_{mn} \beta_{ij,mn}^{cd} \beta_{k\ell,ab}^{mn}.$$

In other words, if and only if

$$\sum_{mn} Q_{jk}\alpha_{ik}^{m}\alpha_{j\ell}^{n}Q_{na}\alpha_{ma}^{c}\alpha_{nb}^{d} = \sum_{mn} Q_{jm}\alpha_{im}^{c}\alpha_{jn}^{d}Q_{\ell a}\alpha_{ka}^{m}\alpha_{\ell b}^{n}.$$

□

Definition 3.3.6: Q-Structure

Let A be an associative algebra with basis $\{e_i\}_{i\in I}$ and structure coefficients α_{ij}^{k}. A choice of coefficients Q_{ij} for all $i, j \in I$ is called a ***Q*-structure** if

$$\sum_{mn} Q_{jk}\alpha_{ik}^{m}\alpha_{j\ell}^{n}Q_{na}\alpha_{ma}^{c}\alpha_{nb}^{d} = \sum_{mn} Q_{jm}\alpha_{im}^{c}\alpha_{jn}^{d}Q_{\ell a}\alpha_{ka}^{m}\alpha_{\ell b}^{n}.$$

Note: We don't impose a corresponding unitality condition, however some of our examples of Q-structures will have units.

As the defining condition of Q-structures is quite abstruse, it's not obvious that any such structures exist. So we begin with a few examples.

Example 3.3.7: Q-Structure: Ordinary Tensor Product

Let A be an associative algebra, with basis $\{e_i\}_{i\in I}$. If we select $Q_{ij} = 1$ for all i, then the algebra structure on $A \otimes_Q A$ reduces to the ordinary one.

Multiplication: By definition, our product is given by

$$(e_i \otimes e_j)(e_k \otimes e_\ell) = e_i e_k \otimes e_j e_\ell.$$

$\boxed{\text{Associativity:}}$ The necessary condition reduces to

$$\sum_{mn} \alpha_{ik}^m \alpha_{j\ell}^n \alpha_{ma}^c \alpha_{nb}^d = \sum_{mn} \alpha_{im}^c \alpha_{jn}^d \alpha_{ka}^m \alpha_{\ell b}^n,$$

which we can write as

$$(\sum_m \alpha_{ik}^m \alpha_{ma}^c)(\sum_n \alpha_{j\ell}^n \alpha_{nb}^d) = (\sum_m \alpha_{im}^c \alpha_{ka}^m)(\sum_n \alpha_{jn}^d \alpha_{\ell b}^n).$$

That this equation is satisfied follows from the fact that A is associative, and thus $\sum_m \alpha_{ik}^m \alpha_{ma}^c = \sum_m \alpha_{im}^c \alpha_{ka}^m$ and similarly $\sum_n \alpha_{j\ell}^n \alpha_{nb}^d = \sum_n \alpha_{jn}^d \alpha_{\ell b}^n$.

Example 3.3.8: Q-Structure: Tensor Algebra

Let $T[X]$ be the tensor algebra on a set X. For all $x, y \in X$ choose some $Q_{xy} \in k$ and for any monomials $\bar{x} = x_1 \ldots x_m$ and $\bar{y} = y_1 \ldots y_n$, define

$$Q_{\bar{x}\bar{y}} \equiv \prod_{ij} Q_{x_i y_j}.$$

This defines a Q-structure on $T[X]$.

$\boxed{\text{Associativity:}}$ Since the canonical basis of $T[X]$ consists of the monomials of elements of X, the structure coefficients take the simple form

$$\alpha_{\bar{i}\bar{j}}^{\bar{k}} = \delta_{\bar{i}+\bar{j},\bar{k}},$$

where $\bar{i}, \bar{j}, \bar{k}$ refer to some sequence of elements in X, and $\bar{i} + \bar{j}$ is the concatenation of the sequences given by $\bar{i}$ and $\bar{j}$. It follows that

$$\begin{aligned}\sum_{\bar{m}\bar{n}} Q_{\bar{j}\bar{k}}\alpha^{\bar{m}}_{\bar{i}\bar{k}}\alpha^{\bar{n}}_{\bar{j}\bar{\ell}}Q_{\bar{n}\bar{a}}\alpha^{\bar{c}}_{\bar{m}\bar{a}}\alpha^{\bar{d}}_{\bar{n}\bar{b}} &= Q_{\bar{j}\bar{k}}Q_{(\bar{j}+\bar{\ell})\bar{a}}\alpha^{\bar{c}}_{(\bar{i}+\bar{k})\bar{a}}\alpha^{\bar{d}}_{(\bar{j}+\bar{\ell})\bar{b}} \\ &= Q_{\bar{j}\bar{k}}Q_{(\bar{j}+\bar{\ell})\bar{a}}\alpha^{\bar{c}}_{(\bar{i}+\bar{k})\bar{a}}\alpha^{\bar{d}}_{(\bar{j}+\bar{\ell})\bar{b}} \\ &= Q_{\bar{j}\bar{k}}Q_{(\bar{j}+\bar{\ell})\bar{a}}(\delta_{\bar{i}+\bar{k}+\bar{a},\bar{c}})(\delta_{\bar{j}+\bar{\ell}+\bar{b},\bar{d}}).\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{\bar{m}\bar{n}} Q_{\bar{j}\bar{m}}\alpha^{\bar{c}}_{\bar{i}\bar{m}}\alpha^{\bar{d}}_{\bar{j}\bar{n}}Q_{\bar{\ell}\bar{a}}\alpha^{\bar{m}}_{\bar{k}\bar{a}}\alpha^{\bar{n}}_{\bar{\ell}\bar{b}} &= Q_{\bar{j}(\bar{k}+\bar{a})}\alpha^{\bar{c}}_{\bar{i}(\bar{k}+\bar{a})}\alpha^{\bar{d}}_{\bar{j}(\bar{\ell}+\bar{b})}Q_{\bar{\ell}\bar{a}} \\ &= Q_{\bar{j}(\bar{k}+\bar{a})}Q_{\bar{\ell}\bar{a}}(\delta_{\bar{i}+\bar{k}+\bar{a},\bar{c}})(\delta_{\bar{j}+\bar{\ell}+\bar{b},\bar{d}})\end{aligned}$$

It follows that we need

$$Q_{\bar{j}\bar{k}}Q_{(\bar{j}+\bar{\ell})\bar{a}} = Q_{\bar{j}(\bar{k}+\bar{a})}Q_{\bar{\ell}\bar{a}}$$

Using $Q_{\bar{x}\bar{y}} \equiv \prod_{ij} Q_{x_i y_j}$, we have $Q_{(\bar{j}+\bar{\ell})\bar{a}} =$

$$\begin{aligned}Q_{(\bar{j}+\bar{\ell})\bar{a}} &= \prod_{rt} Q_{(j+\ell)_r a_t} = \prod_{rst} Q_{j_r a_t}Q_{\ell_s a_t} \\ &= Q_{\bar{j}\bar{a}}Q_{\bar{\ell}\bar{a}}\end{aligned}$$

Thus we have $Q_{\bar{j}\bar{k}}Q_{(\bar{j}+\bar{\ell})\bar{a}} = Q_{\bar{j}\bar{k}}Q_{\bar{j}\bar{a}}Q_{\bar{\ell}\bar{a}} = Q_{\bar{j}(\bar{k}+\bar{a})}Q_{\bar{\ell}\bar{a}}$.

Definition 3.3.9: Q-Bialgebra

A ***Q*-bialgebra** $(B, \{e_i\}_{i\in I}, Q, \Delta, \varepsilon)$ is an associative k-algebra B, with chosen basis $\{e_i\}$, and Q-structure Q_{ij}, along with linear maps $\Delta : B \to B \otimes_Q B$ and $\varepsilon : B \to k$, such that Δ is coassociative and

$$\Delta(xy) = \Delta(x)\Delta(y)$$
$$\varepsilon_1\Delta = \varepsilon_2\Delta = id.$$

Ordinary bialgebras: If $Q_{ij} = 1$ for all i, j, the product on $B \otimes_Q B$ reduces to the ordinary product, and thus B is a Q-bialgebra if and only if B is an ordinary bialgebra in the sense of Example 3.1.3.

Example 3.3.10: Q-Bialgebra: Deformed Tensor Algebra

Let A be a set of elements and Q_{xy} be elements of a ring k, for all $x, y \in A$. Then the tensor algebra $T[A]$ has the structure of a Q-bialgebra where $\Delta a = a \otimes 1 + 1 \otimes a \in T[A] \otimes_Q T[A]$ for all $a \in A$, and

$$\Delta(xy) = \Delta(x)\Delta(y) = \sum Q_{x_{(2)}y_{(1)}} x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)}. \tag{3.46}$$

Interpretation: We can think of this as a tensor bialgebra with a *deformed comultiplication* — the multiplication internal to $T[A]$ remains the ordinary multiplication in the tensor algebra, but the comultiplication is deformed by the condition that Δ must be multiplicative on $T[A] \otimes_Q T[A]$.

$\boxed{\text{Anyons:}}$ Consider the tensor algebra on two elements $\{x, y\}$, and let $Q_{xy} = e^{i\phi}$. Then $\Delta x = 1 \otimes x + x \otimes 1$ and $\Delta y = 1 \otimes y + y \otimes 1$ are still primitive elements. However,

$$\Delta(xy) = xy \otimes 1 + 1 \otimes xy + x \otimes y + e^{i\phi} y \otimes x \tag{3.47}$$

is adjusted by the anyonic commutation condition.

In what follows we will consider graded Q-bialgebras. That is, our bialgebras will have a grading $B = \bigoplus B_n$, through which we can view our elements as bosonic or fermionic, and they will also have the freedom to have some extra non-trivial commutation conditions. This brings up an important question: which tensor product should we use when dealing with these graded Q-bialgebras? We will consider the following interpretation:

$\boxed{\text{Interpretation:}}$ The graded tensor product and Q-tensor product will correspond to *external* and *internal* structure, respectively. The graded tensor product will appear in our discussion of dual spaces, and we will think of it as something like the structure that the linear functionals couple to. On the other hand, the role of the Q-tensor product is to permit certain deformations internal to the bialgebra. To make this more explicit, consider the following example:

Example 3.3.11

Take the exterior algebra on two generators $\Lambda[x, y]$, viewed as a graded algebra with x, y in degree 1, and take a non-trivial $Q_{xy} \in k$. Consider the dual space $\Lambda[x, y]^*$.

$\boxed{\text{External structure:}}$ The convolution product on $\Lambda[x,y]^*$ is defined using the graded tensor product, and so has the form

$$\phi * \psi(w) = \sum (-1)^{|w_{(1)}||\psi|}\phi(w_{(1)})\psi(w_{(2)}),$$

which couples together the graded structures of $\Lambda[x,y]$ and $\Lambda[x,y]^*$ in the coefficient $(-1)^{|w_{(1)}||\psi|}\phi(w_{(1)})$. However, the internal structure doesn't couple to functionals, and instead only appears when we act by linear functions on products in $\Lambda[x,y]$.

$\boxed{\text{Internal structure:}}$ In other words, the Q-structure appears up due to the *internal coupling* of terms inside of $\Lambda[x,y]$. For example

$$\phi * \psi(x) = \phi(1)\psi(x) + (-1)^{|x||\psi|}\phi(x)\psi(1)$$

but

$$\begin{aligned}(\phi * \psi)(xy) &= \phi(1)\psi(xy) + (-1)^{|xy||\psi|}\phi(xy)\psi(1) + (-1)^{|x||\psi|}\phi(x)\psi(y) \\ &\quad + Q_{xy}(-1)^{|y||\psi|}\phi(y)\psi(x) \\ &= \phi(1)\psi(xy) + \phi(xy)\psi(1) + (-1)^{|\psi|}\left(\phi(x)\psi(y) + Q_{xy}\phi(y)\psi(x)\right).\end{aligned}$$

Proposition 3.3.12: Comultiplication on Primitive Elements

Let B be a graded Q-bialgebra and let P be its set of primitive elements. If $\{p_i\}$ is a collection of primitive elements which satisfy $(1 \otimes p_i)(p_j \otimes 1) = Q_{ij} p_j \otimes p_i$ for $i < j$, then

$$\Delta(p_1^{\alpha_1} \dots p_n^{\alpha_n}) = \sum_{k_1,\dots,k_n} \prod_{\substack{i,j=1 \\ i<j}}^{n} \binom{\alpha_i}{k_i}_{q_i} Q_{ij}^{(\alpha_i - k_i)k_j} p_1^{k_1} \dots p_n^{k_n} \otimes p_1^{\alpha_1 - k_1} \dots p_n^{\alpha_n - k_n} \tag{3.48}$$

where $q_i = (-1)^{|p_i|}$.

Proof. This is a straightforward consequence of our multivariable q-Binomial coefficient formula. We apply Lemma 3.3.2 to the algebra $B \otimes_Q B$ and the elements $1 \otimes p_i$ and $p_j \otimes 1$. $\square$

Note: This gives us an explicit form for the comultiplication of every element of Q-bialgebras like $T[X] \otimes_Q T[X]$ which are generated by products and linear combinations of primitive elements. In particular, deformations of symmetric algebras and universal enveloping algebras fall under this umbrella.

3.4 Dual Hopf and Bialgebras

When working with infinite-dimensional algebras, it is often too unwieldy to work with the entire algebraic dual space. An easier way of handling duals in this case is to work with *dual pairings*:

Definition 3.4.1: Dual Pairing of Bialgebras

A **dual pairing** of graded bialgebras $(A, B, \langle\cdot,\cdot\rangle)$ is a dual pairing of A and B as graded vector spaces, such that

$$\begin{aligned} \langle x, yz\rangle &= \sum(-1)^{|y||x_{(2)}|}\left\langle x_{(1)}, y\right\rangle\left\langle x_{(2)}, z\right\rangle \\ \langle xy, z\rangle &= \sum(-1)^{|y||z_{(1)}|}\left\langle x, z_{(1)}\right\rangle\left\langle y, z_{(2)}\right\rangle \\ \langle 1, x\rangle &= \langle x, 1\rangle = \varepsilon(x). \end{aligned} \tag{3.49}$$

$\boxed{\text{Terminology:}}$ The above pairing shows up under different names in the literature. We provide a brief history of terms in Section 3.4.1 below. For our purposes, we shall refer to the above as **Laplace pairing** (the motivation of the name being the pairing's relationship to Laplace formula for determinants, as mentioned in the history below) and will focus on instances of self-pairing: $\langle\cdot,\cdot\rangle : A \otimes A \to k$.

$\boxed{\text{Duals:}}$ The injection $A \to B^*$, $x \to \langle x, -\rangle$, exhibits A as a well-behaved dual of B. Similarly, B can be seen as a nice dual of A.

$\boxed{\text{Hopf Algebras:}}$ If A and B are Hopf algebras, we further require

$$\langle S_A\phi, h\rangle = \langle \phi, S_B h\rangle, \tag{3.50}$$

where S is the antipode.

$\boxed{\text{Dual notion:}}$ The Laplace pairing condition can be written in terms of the multiplication and comultiplication as

$$\begin{aligned} \langle\cdot,\cdot\rangle \circ 1\otimes\mu &= \mu_k(\langle\cdot,\cdot\rangle\otimes\langle\cdot,\cdot\rangle)(1\otimes\sigma\otimes 1)(\Delta\otimes 1\otimes 1) \\ \langle\cdot,\cdot\rangle \circ \mu\otimes 1 &= \mu_k(\langle\cdot,\cdot\rangle\otimes\langle\cdot,\cdot\rangle)(1\otimes\sigma\otimes 1)(1\otimes 1\otimes\Delta) \\ \langle\cdot,\cdot\rangle\, \eta\otimes 1 &= \langle\cdot,\cdot\rangle\, 1\otimes\eta = \varepsilon, \end{aligned} \tag{3.51}$$

where $\sigma : A\otimes B\to B\otimes A$ is the swap map $\sigma(x\otimes y) = (-1)^{|x||y|}y\otimes x$ and μ_k is the multiplication on the algebra k. The dual notion to the Laplace pairing has found great interest. If we dualize the equations above we find the conditions

$$\begin{aligned} 1\otimes\mu^*\circ R &= (\Delta^*\otimes 1\otimes 1)(1\otimes\sigma\otimes 1)(R\otimes R)\mu_k^* \\ \mu^*\otimes 1\circ R &= (1\otimes 1\otimes\Delta^*)(1\otimes\sigma\otimes 1)(R\otimes R)\mu_k^* \\ \eta^*\otimes 1\circ R &= 1\otimes\eta^*\circ R = \varepsilon^*, \end{aligned} \tag{3.52}$$

where $R : k\to H\otimes H$, Δ^* is a multiplication map, μ^* is a comultiplication, η^* is a counit and ε^* a unit. These conditions define a **quasi-triangular structure**. For more information see [30, Chapter 12].

3.4.1 A Brief History

Dual pairings of bialgebras have shown up independently, under various names and guises, throughout the literature. We provide a brief outline of some of the terminology used to describe them.

$\boxed{\text{1974:}}$ The Laplace pairing is introduced in [10].

$\boxed{\text{1986:}}$ Drinfel'd's ICM address considers the dual notion to Hopf algebras with Laplace pairings, under the name **quantum groups** [11].

$\boxed{\text{1991:}}$ Majid calls Hopf algebras with Laplace pairings **dual quasitriangular** Hopf algebras [25].

Nowadays it is common to hear them referred to as **coquasitriangular** Hopf algbras [31].

$\boxed{\text{1991:}}$ Larson uses Laplace pairings to study infinite dimensional quasitriangular bialgebras without having to worry about topological completions of tensor products [22].

$\boxed{\text{1994:}}$ Fischman refers to Laplace pairings as a **braiding** and calls the corresponding Hopf algebra a **cotriangular** Hopf algebra [14].

$\boxed{\text{2001:}}$ Borcherds considers Hopf algebras with Laplace pairing in the guise of **bicharacters** on the group algebra kG [7].

$\boxed{\text{2002:}}$ Brouder uses the term **Laplace pairing** and uses it to derive a version of Wick's theorem [9].

3.4.2 Circle Products

Laplace pairings provide a way of deforming the multiplication on bialgebras. The new multiplication induced by the Laplace pairing is called a circle product, and comes in different flavours, depending on how we use the Laplace pairing to pair together elements:

Definition 3.4.2: Graded Circle Products

Let B be a graded bialgebra with Laplace pairing $\langle \cdot, \cdot \rangle$. The **first circle product** and **second circle product** on B with respect to $\langle \cdot, \cdot \rangle$ are

$$\begin{aligned} x \circ_1 y &= \sum (-1)^{|x_{(2)}||y_{(1)}|} \left\langle x_{(1)}, y_{(1)} \right\rangle x_{(2)} y_{(2)}. \\ x \circ_2 y &= \sum (-1)^{|x_{(2)}||y_{(1)}|} \left\langle x_{(2)}, y_{(2)} \right\rangle x_{(1)} y_{(1)}. \end{aligned} \tag{3.53}$$

$\boxed{\text{Note:}}$ In the case that B is graded cocommutative $\circ_1 = \circ_2$.

Lemma 3.4.3

Let B be a bialgebra with Laplace pairing $\langle \cdot, \cdot \rangle$. Then

$$\langle x \circ_1 y, z \rangle = \langle x, y \circ_2 z \rangle . \tag{3.54}$$

Proof.

$$\begin{aligned}\langle x \circ_1 y, z \rangle &= \sum (-1)^{|y_{(1)}||x_{(2)}|} \left\langle x_{(1)}, y_{(1)} \right\rangle \left\langle x_{(2)} y_{(2)}, z \right\rangle \\ &= \sum (-1)^{|y_{(1)}||x_{(2)}| + |z_{(1)}||y_{(2)}|} \left\langle x_{(1)}, y_{(1)} \right\rangle \left\langle x_{(2)}, z_{(1)} \right\rangle \left\langle y_{(2)}, z_{(2)} \right\rangle .\end{aligned}$$

$$\begin{aligned}\langle x, y \circ_2 z \rangle &= \sum (-1)^{|z_{(1)}||y_{(2)}|} \left\langle y_{(2)}, z_{(2)} \right\rangle \left\langle x, y_{(1)} z_{(1)} \right\rangle \\ &= \sum (-1)^{|y_{(1)}||x_{(2)}| + |z_{(1)}||y_{(2)}|} \left\langle y_{(2)}, z_{(2)} \right\rangle \left\langle x_{(1)}, y_{(1)} \right\rangle \left\langle x_{(2)}, z_{(1)} \right\rangle \\ &= \langle x \circ_1 y, z \rangle .\end{aligned}$$

□

Thus the circle products behave something like Frobenius algebra multiplications, with respect to the Laplace pairing. In the case that our Hopf or bialgebra is cocommutative, we will have $\circ_1 = \circ_2$. Standard results for the circle product on cocommutative bialgebras can be found in [9]. We quote two such results below.

Lemma 3.4.4

Let B be a cocommutative bialgebra with Laplace pairing $\langle \cdot, \cdot \rangle$. The induced circle product is unital with unit 1 (the unit in the bialgebra under its usual multiplication).

Lemma 3.4.5

Let B be a cocommutative bialgebra with Laplace pairing $\langle \cdot, \cdot \rangle$. Then

$$\langle x, y \rangle = \varepsilon(x \circ y)$$

for all x, y.

Proof. See [9, Lemma 2.6]. □

Primitive and group-like elements are both cocommutative and provide a entry point into looking at how elements of bialgebras behave with respect to the circle product. We will next look at how Laplace pairings provide a way of recovering Wick's theorem, a standard result in quantum field theory. The following results can be found for the bosonic case in [9], but we provide a generalization to the setting of Q-bialgebras.

$\boxed{\text{Notation:}}$ In the following proofs, the notation $\bar{q} \equiv q_1 \ldots q_n$ and $\bar{q}^\circ \equiv q_1 \circ_1 \ldots \circ_1 q_n$ will prove useful in simplifying equations. Similarly, we shall use

$$\begin{aligned} \bar{q}_{\neq i} &= q_1 \ldots q_{i-1} q_{i+1} \ldots q_n \\ \bar{q}^\circ_{\neq i} &= q_1 \circ_1 \ldots \circ_1 q_{i-1} \circ_1 q_{i+1} \circ_1 \ldots \circ_1 q_n . \end{aligned}$$

If $i_1, \ldots, i_n$ are multiple terms to be removed from a product, we shall use $\bar{q}_{\neq i_1,\ldots,i_n}$ or $\bar{q}_{\neq i}$, when it's clear that i represents multiple indices.

Theorem 3.4.6: Wick's Theorem

Let B be a graded Q-bialgebra with Laplace pairing $\langle \cdot, \cdot \rangle$, and let $q_1, \ldots, q_m, p_1, \ldots, p_n$ be primitive elements, and $q_i q_j = Q_{ij} q_j q_i$. Then

$$\langle q_1 \ldots q_m, p_1 \ldots p_n \rangle = \sum_{\sigma \in S_n} \prod_{i=1}^{n} \prod_{\substack{a,b=1 \\ \sigma(a)<\sigma(i) \\ i<a,b}}^{n} Q_{\sigma(a),\sigma(i)} (-1)^{|p_i||q_{\sigma(b)}|} \left\langle q_{\sigma(i)}, p_i \right\rangle \tag{3.55}$$

if $m = n$, and is zero otherwise.

Proof. First note that $\langle 1, p_1 \ldots p_n \rangle = \langle 1, p_1 \rangle p_2 \ldots p_n = 0$ for any $p_i \in P$.

$\boxed{\langle q, p_1 \ldots p_n \rangle = 0:}$ Let $q, p_1, \ldots, p_n \in P$ with $n \geq 2$. Applying the Laplace pairing condition, we have

$$\begin{aligned}
\langle q, p_1 \ldots p_n \rangle &= \sum (-1)^{|q_{(2)}||p_1|} \left\langle q_{(1)}, p_1 \right\rangle \left\langle q_{(2)}, p_2 \ldots p_n \right\rangle \\
&= (-1)^{|q||p_1|} \langle 1, p_1 \rangle \langle q, p_2 \ldots p_n \rangle + \langle q, p_1 \rangle \langle 1, p_2 \ldots p_n \rangle \\
&= 0.
\end{aligned}$$

Similarly

$$\langle q_1 \ldots q_n, p \rangle = (-1)^{|p||q_n|} \langle q_1, p \rangle \langle q_2 \ldots q_n, 1 \rangle + \langle q_1, 1 \rangle \langle q_2 \ldots q_n, p \rangle = 0.$$

$\boxed{\text{In General:}}$ We apply Proposition 3.3.12 to write

$$\Delta(q_1 \ldots q_m) = \sum_{k_1, \ldots, k_m} \prod_{i,j=1}^{m} \binom{1}{k_i}_{Q_i} Q_{ij}^{(1-k_i)k_j} q_1^{k_1} \ldots q_m^{k_m} \otimes q_1^{1-k_1} \ldots q_{nm}^{1-k_m}. \tag{3.56}$$

Since each k_i is either zero or one, each $\binom{1}{k_i}_{Q_i} = 1$. Using the defining property of the Laplace pairing, we can comultiply the first component in $\langle \bar{q}, \bar{p} \rangle$ to write

$$\langle q_1 \dots q_m, p_1 \dots p_n \rangle = \sum_{k_1,\dots,k_m} \prod_{i,j=1}^{m} Q_{ij}^{(1-k_i)k_j} (-1)^{|p_1||q_1^{1-k_1}| + \dots |p_1||q_m^{1-k_m}|} \times \left\langle q_1^{k_1} \dots q_m^{k_m}, p_1 \right\rangle \left\langle q_1^{1-k_1} \dots q_m^{1-k_m}, p_2 \dots p_n \right\rangle .$$

Now $\left\langle q_1^{k_1} \dots q_m^{k_m}, p_1 \right\rangle$ will go to zero unless there is precisely one factor on the left, say q_j, in which case every other q_k will appear on the left side of the Laplace pairing $\left\langle q_1^{1-k_1} \dots q_m^{1-k_m}, p_2 \dots p_n \right\rangle$. So we can change our sum over $k_1, \dots, k_m$ to one over a single variable k which will pick out each k_i to set to 1. For each such k, the only Q_{ij} that will appear is of the form Q_{ik}. We can also combine the powers of (-1), noting that $|x_1 \dots x_n| = |x_1| + \dots + |x_n|$. Thus we have

$$\langle q_1 \dots q_m, p_1 \dots p_n \rangle = \sum_{k=1}^{m} \prod_{i=1}^{m} Q_{ik} (-1)^{|p_1||\bar{q}_{\neq k}|} \langle q_k, p_1 \rangle \langle \bar{q}_{\neq k}, p_2 \dots p_n \rangle .$$

$\boxed{\text{Induction:}}$ We can apply induction to $\langle \bar{q}_{\neq k}, \bar{p}_{\neq 1} \rangle$. In particular, we are non-zero if and only if the length of $\bar{q}_{\neq k}$ and $\bar{p}_{\neq 1}$ are equal. That is $m - 1 = n - 1$. So the only non-zero possibility occurs when $m = n$. So we're computing

$$\langle q_1 \dots q_n, p_1 \dots p_n \rangle = \sum_{k=1}^{n} \prod_{i=1}^{n} Q_{ik} (-1)^{|p_1||\bar{q}_{\neq k}|} \langle q_k, p_1 \rangle \langle \bar{q}_{\neq k}, \bar{p}_{\neq 1} \rangle .$$

Next note that the induction hypothesis

$$\langle q_1 \dots q_r, p_1 \dots p_r \rangle = \sum_{\sigma \in S_t} \prod_{i=1}^{t} \prod_{\substack{a,b=1 \\ \sigma(a)<\sigma(i) \\ i<a,b}}^{t} Q_{\sigma(a),\sigma(i)} (-1)^{|p_i||q_{\sigma(b)}|} \left\langle q_{\sigma(i)}, p_i \right\rangle$$

uses the terms $q_1, \dots, q_r$ and $p_1, \dots, p_r$, rather than $\{q_1, \dots q_n\}_{\neq k}$ or $p_2, \dots, p_n$. So we first define an **order-preserving** bijection $\phi : \{i\}_{i=1}^{n-1} \to \{i\}_{i=1}^{n} \setminus \{k\}$ by

$$\bar{q}_{\neq k} = q_{\phi(1)} \dots q_{\phi(n-1)}.$$

Then

$$\begin{aligned} \langle \bar{q}_{\neq k}, \bar{p}_{\neq 1} \rangle &= \left\langle q_{\phi(1)} \dots q_{\phi(n-1)}, p_1 \dots p_{n-1} \right\rangle \\ &= \sum_{\sigma \in S_{n-1}} \prod_{j=1}^{n-1} \prod_{\substack{a,b=1 \\ \sigma(a)<\sigma(j) \\ j<a,b}}^{n-1} Q_{\phi(\sigma(a)),\phi(\sigma(j))} (-1)^{|p_{j+1}||q_{\phi(\sigma(b))}|} \left\langle q_{\phi(\sigma(j))}, p_{j+1} \right\rangle . \end{aligned}$$

So we have

$$\begin{aligned} \langle \bar{q}, \bar{p} \rangle = &\sum_{k=1}^{n} \prod_{i=1}^{n} Q_{ik} (-1)^{|p_1||\bar{q}_{\neq k}|} \langle q_k, p_1 \rangle \\ &\times \sum_{\sigma \in S_{n-1}} \prod_{j=1}^{n-1} \prod_{\substack{a,b=1 \\ \sigma(a)<\sigma(j) \\ j<a,b}}^{n-1} Q_{\phi(\sigma(a)),\phi(\sigma(j))} (-1)^{|p_{j+1}||q_{\phi(\sigma(b))}|} \left\langle q_{\phi(\sigma(j))}, p_{j+1} \right\rangle . \end{aligned}$$

$\boxed{\text{Simplifying:}}$ To simplify the above formula, first note that $\phi \circ \sigma$ accounts for all permutations of $\{1, \dots, n\} \setminus \{k\}$. Together with the sum over k of $\langle q_k, p_1\rangle$, all permutations of $\{1,\ldots,n\}$ are accounted for. Thus

$$\langle \bar{q}, \bar{p}\rangle = \sum_{\sigma\in S_n}\prod_{i=1}^{n}\alpha(\sigma,i)\left\langle q_{\sigma(i)}, p_i\right\rangle$$

where $\alpha(\sigma, i)$ are some coefficients. Next we check these coefficients. The overall (-1) factor, for a term with fixed k and σ, will be

$$\prod_{\substack{i,b=1\\ i<b}}^{n-1}(-1)^{|p_1||\bar{q}_{\neq k}|+|p_{i+1}||q_{\phi(b)}|} = \prod_{\substack{i,b=1\\ i<b}}^{n-1}\prod_{\substack{j=1\\ j\neq k}}^{n}(-1)^{|p_1||q_j|+|p_{i+1}||q_{\phi(\sigma(b))}|}.$$

Let σ' be the permutation of S_n defined by $\sigma'(1) = k$, $\sigma'(i) = \phi(\sigma(i-1))$ for $i > 1$. Note that $j \neq k$ above if and only if $j = \phi(\ell)$ for some $1 \leq \ell \leq n-1$, and hence $j = \phi(i-1)$ for some $2 \leq i \leq n$. Since we pick up all such i, and multiplication of scalars is commutative, the permutation σ is irrelevant, so we can write

$$\begin{aligned}\prod_{\substack{i,b=1\\ i<b}}^{n-1}\prod_{\substack{j=1\\ j\neq k}}^{n}(-1)^{|p_1||q_j|+|p_{i+1}||q_{\phi(\sigma(b))}|} &= \prod_{\substack{b=1\\ 1<b}}^{n}(-1)^{|p_1||q_{\sigma'(b)}|}\prod_{\substack{i,b=2\\ i<b}}^{n}(-1)^{|p_i||_{\phi(\sigma(b-1))}|}\\ &= \prod_{\substack{b=1\\ i<b}}^{n}(-1)^{|p_i||q_{\sigma'(b)}|}\end{aligned}$$

as required.

$\boxed{Q \text{ terms:}}$ Finally we check the Q coefficients. The overall Q factor will be

$$\prod_{i=1}^{n} \prod_{\substack{a,j=1 \\ \sigma(a)<\sigma(j) \\ j<a}}^{n-1} Q_{ik} Q_{\phi(\sigma(a)),\phi(\sigma(j))} = \prod_{i=1}^{n} \prod_{\substack{a,j=1 \\ \sigma(a)<\sigma(j) \\ j<a}}^{n-1} Q_{i,\sigma'(1)} Q_{\sigma'(a+1),\sigma'(j+1)},$$

where we used the fact that, by convention $Q_{ij} = 1$ for $j \leq i$. Since ϕ is ordering-preserving, $\sigma(a) < \sigma(j)$ if and only if $\phi(\sigma(a)) < \phi(\sigma(j))$. In other words, if and only if $\sigma'(a+1) < \sigma'(j+1)$. We then have

$$\prod_{i=1}^{\sigma'(1)-1} \prod_{\substack{a,j=2 \\ \sigma'(a)<\sigma'(j) \\ j<a}}^{n} Q_{i,\sigma'(1)} Q_{\phi(a-1),\sigma'(j)}.$$

Note that if $j = 1$, $j < a$ is always satisfied, and $\sigma'(a) < \sigma'(1) = k$ is satisfied for all a such that $i = \sigma'(a) < k$, which is precisely what we acquire in our $Q_{i,\sigma'(1)}$ term. Thus, the above expression is equivalent to

$$\prod_{\substack{a,j=1 \\ \sigma'(a)<\sigma'(j) \\ j<a}}^{n} Q_{\sigma'(a),\sigma'(j)}$$

and we acquire Wick's theorem. $\square$

$\boxed{\text{Intuition:}}$ The above formulation of Wick's theorem is rather obscure, so below we consider a few small examples.

Example 3.4.7: Wick's Theorem: $n = 2$

Consider a graded Q-bialgebra with Laplace pairing. We have

$$\langle q_1 q_2, p_1 p_2 \rangle = (-1)^{|p_1||q_2|} \langle q_1, p_1 \rangle \langle q_2, p_2 \rangle + Q_{12} (-1)^{|p_1||q_1|} \langle q_2, p_1 \rangle \langle q_1, p_2 \rangle$$

Below we break into cases of each permutation of S_2, applying the formula

$$\langle q_1 \ldots q_n, p_1 \ldots p_n \rangle = \sum_{\sigma \in S_n} \prod_{i=1}^{n} \prod_{\substack{a,b=1 \\ \sigma(a)<\sigma(i) \\ i<a,b}}^{n} Q_{\sigma(a),\sigma(i)} (-1)^{|p_i||q_{\sigma(b)}|} \left\langle q_{\sigma(i)}, p_i \right\rangle .$$

$\boxed{(12):}$ Here $\sigma(i) = i$ for all i, so there are no a, i satisfying $i < a$, $\sigma(a) < \sigma(i)$. Hence there are no Q terms. On the other hand, since $1 < 2$, we have a factor of $(-1)^{|p_1||q_2|}$:

$$(-1)^{|p_1||q_2|} \langle q_1, p_1 \rangle \langle q_2, p_2 \rangle .$$

$\boxed{(21):}$ We have $1 < 2$ and $\sigma(1) = 2 > \sigma(2) = 1$, so we pick up a factor of Q_{12}. And in similar fashion to the previous case, $1 < 2$ gives us a (-1) factor. So we have

$$Q_{12}(-1)^{|p_1||q_1|} \langle q_2, p_1 \rangle \langle q_1, p_2 \rangle .$$

Example 3.4.8: Wick's Theorem: $n = 3$

Wick's theorem gives

$$\begin{aligned}
\langle q_1 q_2 q_3, p_1 p_2 p_3 \rangle = \alpha_{123} & \langle q_1, p_1 \rangle \langle q_2, p_2 \rangle \langle q_3, p_3 \rangle + \alpha_{132} \langle q_1, p_1 \rangle \langle q_3, p_2 \rangle \\
& + \alpha_{213} \langle q_2, p_1 \rangle \langle q_1, p_2 \rangle \langle q_3, p_3 \rangle + \alpha_{231} \langle q_2, p_1 \rangle \langle q_3, p_2 \rangle \langle q_1, p_3 \rangle \\
& + \alpha_{312} \langle q_3, p_1 \rangle \langle q_1, p_2 \rangle \langle q_2, p_3 \rangle + \alpha_{321} \langle q_3, p_1 \rangle \langle q_2, p_2 \rangle \langle q_1, p_3 \rangle .
\end{aligned}$$

For the sake of space, we list out the explicit values of the coefficients below.

We consider each permutation in turn. In all cases below, we pick up (-1) factors corresponding to $1 < 2$, $1 < 3$, and $2 < 3$:

$$(-1)^{|p_1|(|q_{\sigma(2)}|+|q_{\sigma(3)}|)+|p_2||q_{\sigma(3)}|}.$$

$\boxed{(123):}$ Since σ is trivial there is no Q term.

$$\alpha_{123} = (-1)^{|p_1|(|q_2|+|q_3|)+|p_2||q_3|} \langle q_1, p_1 \rangle \langle q_2, p_2 \rangle \langle q_3, p_3 \rangle .$$

$\boxed{(132):}$ $2 < 3$ and $\sigma(2) = 3 > \sigma(3) = 2$ giving Q_{23}.

$$\alpha_{132} = Q_{23}(-1)^{|p_1|(|q_3|+|q_2|)+|p_2||q_2|} \langle q_1, p_1 \rangle \langle q_3, p_2 \rangle \langle q_2, p_3 \rangle .$$

$\boxed{(213):}$ $\sigma(1) > \sigma(2)$ gives Q_{12}:

$$\alpha_{213} = Q_{12}(-1)^{|p_2|(|q_1|+|q_3|)+|p_1||q_3|} \langle q_2, p_1 \rangle \langle q_1, p_2 \rangle \langle q_3, p_3 \rangle .$$

$\boxed{(231):}$ $\sigma(1), \sigma(2) > \sigma(3)$:

$$\alpha_{231} = Q_{13}Q_{23}(-1)^{|p_2|(|q_3|+|q_1|)+|p_3||q_1|} \langle q_2, p_1 \rangle \langle q_3, p_2 \rangle \langle q_1, p_3 \rangle .$$

$\boxed{(312):}$ $\sigma(1) > \sigma(2), \sigma(3)$:

$$\alpha_{312} = Q_{12}Q_{13}(-1)^{|p_3|(|q_1|+|q_2|)+|p_1||q_2|} \langle q_3, p_1 \rangle \langle q_1, p_2 \rangle \langle q_2, p_3 \rangle .$$

$\boxed{(321):}$ $\sigma(1) > \sigma(2) > \sigma(3)$:

$$\alpha_{321} = Q_{12}Q_{13}Q_{23}(-1)^{|p_3|(|q_2|+|q_1|)+|p_2||q_1|} \langle q_3, p_1 \rangle \langle q_2, p_2 \rangle \langle q_1, p_3 \rangle .$$

Example 3.4.9: Wick's Theorem: Ungraded Ordinary Bialgebra

In the case of trivial grading and Q-structure, there are no Q terms nor factors of (-1). Wick's theorem takes the simpler form

$$\langle q_1 \dots q_n, p_1 \dots p_n \rangle = \sum_{\sigma \in S_n} \prod_{i=1}^{n} \left\langle q_{\sigma(i)}, p_i \right\rangle .$$

In other words, Laplace pairings of products of primitive elements can be computed by taking all possible pairs of the terms in the products. In quantum field theory, correlation functions are computed on field operators by taking the vacuum expectation values of time ordered products of fields: $\langle 0 \mid \mathcal{T}\phi(x_1) \dots \phi(x_n) \mid 0 \rangle$. Wick's theorem in that context is the statement that such correlation functions can be computed by taking the sum of the product of all 2-point functions $\langle 0 \mid \mathcal{T}\phi(x_i)\phi(x_j) \mid 0 \rangle$.

As a corollary, we can immediately characterize all possible Laplace pairings on tensor algebras, exterior algebras and symmetric algebras.

Proposition 3.4.10: Laplace Pairings on Tensor-Like Algebras

Let H be a symmetric, exterior, or tensor algebra on a vector space of generators X. Then any bilinear function $\langle\cdot,\cdot\rangle : X \times X \to k$ defines a unique Laplace pairing, whose value on arbitrary elements is given by

$$\langle q_1 \dots q_m, p_1 \dots p_n\rangle = \begin{cases} 0, & \text{if } m \neq n, \\ \sum_{\sigma\in S_n} \prod_{i=1}^{n} \prod^{n}_{\substack{a,b=1 \\ \sigma(a)<\sigma(i) \\ i<a,b}} Q_{\sigma(a),\sigma(i)}(-1)^{|p_i||q_{\sigma(b)}|} \left\langle q_{\sigma(i)}, p_i\right\rangle & \text{if } m = n \end{cases}$$

for $q_i, p_i \in X$.

Next we look at how the Laplace pairing acts on circle products. It turns out that circle products of primitive elements have a simple structure, allowing us to acquire another version of Wick's theorem.

$\boxed{\text{Note:}}$ For the sake of space in what follows, we shall use $\circ$ to denote the **first circle product**, unless otherwise noted.

Lemma 3.4.11: Circle Product of Primitive Elements

Let $p_1, \dots, p_n$ be primitive elements in a Q-bialgebra such that $(1 \otimes p_i)(p_j \otimes 1) = Q_{ij} p_j \otimes p_i$. Then

$$\bar{p}^\circ = \sum_{m=0}^{\lfloor n/2\rfloor} \sum_{\substack{i_1<\dots<i_m \\ j_1,\dots,j_m \\ i_t<j_t,\forall t \\ I\cap J=\varnothing}} \prod_{t=1}^{m} \langle p_{i_t}, p_{j_t}\rangle \prod_{\substack{b\in I,c\in J \\ a,d\notin I\cup J \\ a<b}} Q_{ab}(-1)^{|p_c||p_d|}\bar{p}_{\neq i,j}$$

where $\bar{p}^\circ = (\dots(p_1 \circ p_2)\circ\dots\circ p_{n-1})\circ p_n$ and $\circ$ is the first circle product.

Proof. Recalling Proposition 3.3.12, we have

$$\Delta(p_1 \dots p_n) = \sum_{k_1,\dots,k_n=0}^{1} p_1^{k_1} \dots p_n^{k_n} \otimes p_1^{1-k_1} \dots p_n^{1-k_n} \prod_{\substack{i,j=1\\ i<j}}^{n} Q_{ij}^{(1-k_i)k_j}.$$

Thus

$$\begin{aligned}(p_1 \dots p_n) \circ q_{n+1} = &\sum_{k_1,\dots,k_n=0}^{1} (-1)^{|p_{n+1}||\prod_i p_i^{1-k_i}|} \left\langle p_1^{k_1} \dots p_n^{k_n}, p_{n+1} \right\rangle p_1^{1-k_1} \dots p_n^{1-k_n} \prod_{\substack{i,j=1\\ i<j}}^{n} Q_{ij}^{(1-k_i)k_j} \\ &+ \sum_{k_1,\dots,k_n=0}^{1} \left\langle p_1^{k_1} \dots p_n^{k_n}, 1 \right\rangle p_1^{1-k_1} \dots p_n^{1-k_n} p_{n+1} \prod_{\substack{i,j=1\\ i<j}}^{n} Q_{ij}^{(1-k_i)k_j}.\end{aligned}$$

Using the form of Wick's theorem for the ordinary product (Theorem 3.4.6), this reduces to

$$(p_1 \dots p_n) \circ p_{n+1} = \sum_{k=1}^{n} (-1)^{|p_{n+1}||\prod_{i\neq k} p_i|} \langle p_k, p_{n+1} \rangle \bar{p}_{\neq k} \prod_{\substack{i=1\\ i<k}}^{n} Q_{ik} + p_1 \dots p_n p_{n+1}.$$

We now proceed by induction.

$\boxed{n = 2:}$

$$\begin{aligned}p_1 \circ p_2 &= \sum (-1)^{|p_{1(2)}||p_{2(1)}|} \left\langle p_{1(1)}, p_{2(1)} \right\rangle p_{1(2)} p_{2(2)} \\ &= \langle 1, 1 \rangle p_1 p_2 + (-1)^{|p_2||p_1|} \langle 1, p_2 \rangle p_1 + \langle p_1, 1 \rangle p_2 + \langle p_1, p_2 \rangle 1 \\ &= p_1 p_2 + \langle p_1, p_2 \rangle 1.\end{aligned}$$

$\boxed{\text{Induct:}}$ If the result holds for n, we have

$$(\ldots(p_1 \circ p_2)\ldots \circ p_n) \circ p_{n+1} = \sum_{m=0}^{\lfloor n/2 \rfloor} \sum_{\substack{i_1<\ldots<i_m \\ j_1,\ldots,j_m \\ i_t<j_t,\forall t \\ I\cap J=\varnothing}} \prod_{t=1}^{m} \langle p_{i_t}, p_{j_t}\rangle \prod_{\substack{b\in I, c\in J \\ a,d\notin I\cup J \\ a<b}} Q_{ab}(-1)^{|p_c||p_d|}\bar{p}_{\neq i,j} \circ p_{n+1}$$

$$= \sum\sum\prod\prod Q_{ab}(-1)^{|p_c||p_d|} \langle p_i, p_j\rangle$$
$$\times \left(\sum_{\substack{k=1 \\ k\neq i,j}}^{n} \langle p_k, p_{n+1}\rangle \, \bar{p}_{\neq i,j,k}(-1)^{|p_{n+1}||\prod_{\ell\neq i,j,k} p_\ell|} \prod_{\substack{i=1 \\ i<k}}^{n} Q_{ik} + \bar{p}_{\neq i,j} p_{n+1} \right)$$

where for each choice of $i_1, \ldots, i_m$ and $j_1, \ldots, j_m$, $I = \{i_1, \ldots, i_m\}$ and $J = \{j_1, \ldots, j_m\}$. Expanding the above equation, we have two types of terms:

$$\sum\sum\prod\prod Q_{ab}(-1)^{|p_c||p_d|} \langle p_i, p_j\rangle \sum_{\substack{k=1 \\ k\neq i,j}}^{n} \langle p_k, p_{n+1}\rangle \, \bar{p}_{\neq i,j,k} \prod_{\substack{i=1 \\ i<k}}^{n} Q_{ik}(-1)^{|p_{n+1}||\prod_{\ell\neq i,j,k} p_\ell|}$$
$$+ \sum\sum\prod\prod Q_{ab}(-1)^{|p_c||p_d|} \langle p_i, p_j\rangle \, \bar{p}_{\neq i,j} p_{n+1}.$$

For the first term, note that $\langle p_k, p_{n+1}\rangle$ satisfies the conditions $k < n+1$, $k \notin I$, $n+1 \notin J$. Thus it is one of the terms that will appear in the sum

$$\sum_{\substack{i_1<\ldots<i_{m+2} \\ j_1,\ldots,j_{m+2} \\ i_t<j_t,\forall t \\ I\cap J=\varnothing}} (\ldots).$$

In fact, sum we sum over $k \neq i, j$, all such terms are accounted for. On the other hand, the terms in

$$\sum\sum\prod\prod Q_{ab}(-1)^{|p_c||p_d|}\left\langle p_i,p_j\right\rangle\bar{p}_{\neq i,j}p_{n+1}$$

account for all of the terms that arise when we only pair m terms together in $(\ldots(p_1\circ p_2)\ldots\circ p_n)\circ p_{n+1}$. Next we check that we obtain the correct coefficients.

$\boxed{\text{Coefficients:}}$ For those terms which pair only m p's together, there is no change to the coefficients, since the coefficients depend only on I and J. Thus we obtain the correct coefficients in this case from

$$\sum\sum\prod\prod Q_{ab}(-1)^{|p_c||p_d|}\left\langle p_i,p_j\right\rangle\bar{p}_{\neq i,j}p_{n+1}.$$

On the other hand, for those terms which pair together $m+2$ p's, we need to pick up new coefficients. The Q_{ab} terms that appear will be those with $b\in I$, $a\notin I\cup J$ and $a<b$. Those terms which hadn't already been accounted for are those with $a\notin I\cup J\cup\{k,n+1\}$ and $b=k$, which are precisely the terms picked up by

$$\prod_{\substack{i=1\\i<k}}^{n}Q_{ik}.$$

Finally we check the coefficients of the form (-1). The new terms we need will be $(-1)^{|p_c||p_d|}$ where $c=n+1$ and $d\notin I\cup J\cup\{k\}$. Noting

$$(-1)^{|p_{n+1}||\prod_{\ell\neq i,j,k}p_\ell|}=(-1)^{|p_{n+1}|\sum_{\ell\neq i,j,k}|p_\ell|}=\prod_{\ell\neq i,j,k}(-1)^{|p_{n+1}||p_\ell|}$$

we see that we pick up precisely the necessary terms. Thus by induction, the result holds. $\square$

Example 3.4.12: Circle Product: $n = 3$

$$(p_1 \circ p_2) \circ p_3 = p_1 p_2 p_3 + \langle p_1, p_2 \rangle (-1)^{|p_2||p_3|} p_3 + (-1)^{|p_2||p_3|} \langle p_1, p_3 \rangle p_2 + (-1)^{|p_1|(|p_2|+|p_3|)} \langle p_2, p_3 \rangle p_1.$$

$\boxed{m = 0:}$ $I = J = \varnothing$. Thus there are no Laplace pairing terms, Q terms or (-1) factors. Thus our term is simply of the form

$$p_1 p_2 p_3.$$

$\boxed{m = 2:}$ We have a single pairing. There are three options, since we must have $i_1 < j_1$:

- $\boxed{i_1 = 1, j_1 = 2:}$ We have $b = 1$, $c = 2$. We also have $a, d \in \{1, 2, 3\} \backslash \{1, 2\}$ with $a < b$. Thus $d = 3$ but there is no appropriate a index (and thus no Q term). So our term in this case is

$$\langle p_1, p_2 \rangle (-1)^{|p_2||p_3|} p_3.$$

- $\boxed{i_1 = 1, j_1 = 3:}$ $b = 1, c = 3$, $a, d = 2$, but $a \nless b$, so there is no Q term. Thus we have

$$\langle p_1, p_3 \rangle (-1)^{|p_3||p_2|} p_2.$$

- $\boxed{i_1 = 2, j_1 = 3:}$ $b = 2, c = 3,\ a, d = 1$. Here $a < b$, so we have both a Q term and a (-1) factor:

$$\langle p_2, p_3 \rangle\, Q_{12}(-1)^{|p_3||p_1|} p_1.$$

Proposition 3.4.13: Wick's Theorem for the Circle Product

Let $\{q_i\}_{i=1}^m$, $\{p_j\}_{j=1}^n$ be primitive elements in a Q-bialgebra, with $(1 \otimes q_i)(q_j \otimes 1) = Q_{ij} q_j \otimes q_i$, and $(1 \otimes p_i)(p_j \otimes 1) = P_{ij} p_j \otimes p_i$. Then

$$\langle \bar{q}^\circ, \bar{p}^\circ \rangle = \sum_{m_q=0}^{\lfloor n_q/2 \rfloor} \sum_{\substack{I_q, I_p, J_q, J_p \\ m_p = \frac{1}{2}(n_p - n_q) + m_q}} \prod_{t_q, t_p = 1}^{m_q, m_p} \left\langle q_{i_{t_q}}, q_{j_{t_q}} \right\rangle \left\langle p_{i_{t_p}}, p_{j_{t_p}} \right\rangle$$

$$\times \prod_{\substack{b \in I, c \in J \\ a, c \notin I \cup J \\ a < b}} Q_{a_q b_q} P_{a_p b_p} (-1)^{|q_{c_q}||q_{d_q}| + |p_{c_p}||p_{d_p}|} \left\langle \bar{q}_{\neq i^q, j^q}, \bar{p}_{\neq i^p, j^p} \right\rangle$$

where $a_t, d_t \notin I_t \cup J_t$, $b_t \in I_t, c_t \in J_t$, $a_t < b_t$ with $t \in \{p, q\}$.

Proof. From Lemma 3.4.11 we have

$$\bar{q}^\circ = \sum_{m=0}^{\lfloor n/2 \rfloor} \sum_{\substack{i_1 < \ldots < i_m \\ j_1, \ldots, j_m \\ i_t < j_t, \forall t \\ I \cap J = \varnothing}} \prod_{t=1}^{m} \langle p_{i_t}, p_{j_t} \rangle \prod_{\substack{b \in I, c \in J \\ a, d \notin I \cup J \\ a < b}} Q_{ab} (-1)^{|p_c||p_d|} \bar{p}_{\neq i,j}.$$

Thus we need to compute

$$\langle \bar{q}^\circ, \bar{p}^\circ \rangle = \sum_{m_q=0}^{\lfloor n_q/2 \rfloor} \sum_{m_p=0}^{\lfloor n_p/2 \rfloor} \sum_{\substack{I_q, J_q \\ I_p, J_p}} \prod_{t_q, t_p=1}^{m_q, m_p} \left\langle q_{i_{t_q}}, q_{j_{t_q}} \right\rangle \left\langle p_{i_{t_p}}, p_{j_{t_p}} \right\rangle$$
$$\times \prod_{\substack{b \in I, c \in J \\ a, d \notin I \cup J \\ a < b}} Q_{a_q b_q} P_{a_p b_p} (-1)^{|q_{c_q}||q_{d_q}|} (-1)^{|p_{c_p}||p_{d_p}|} \left\langle \bar{q}_{\neq i^q, j^q}, \bar{p}_{\neq i^p, j^p} \right\rangle, \tag{3.57}$$

which reduces to the computation of $\langle \bar{q}_{\neq i^q, j^q}, \bar{p}_{\neq i^p, j^p} \rangle$, for which we apply Wick's theorem on the ordinary product, Theorem 3.4.6. In particular, the pairing is zero unless the length of the two products in the pairing are equal. The length on the left is $n_q - 2m_q$, and on the right is $n_p - 2m_p$, so we require

$$m_p - m_q = \frac{1}{2}(n_p - n_q).$$

□

Example 3.4.14: Circle Product Wick's Theorem: $n_q, n_p = 2$

$$\langle q_1 \circ q_2, p_1 \circ p_2 \rangle = \langle q_1, q_2 \rangle \langle p_1, p_2 \rangle + (-1)^{|q_2||p_1|} \langle q_1, p_1 \rangle \langle q_2, p_2 \rangle$$
$$+ Q_{12} (-1)^{|q_1||p_1|} \langle q_1, p_2 \rangle \langle q_2, p_1 \rangle .$$

First we note that in this case

$$\begin{aligned} m_p &= \frac{1}{2}(n_p - n_q) + m_q \\ &= m_q. \end{aligned}$$

$\boxed{(m_q, m_p) = (0,0):}$ The only pairings that appear are those of the form $\langle q, p\rangle$. There are two subcases, corresponding to the two ways of ordering $\{q_1, q_2\}$

- $\boxed{(1,2):}$ There are no $\ell, r \in \{1,2\}$ such that $\ell < r$ and $\sigma(r) < \sigma(\ell)$. Consequently, no Q terms appear. On the other hand, we have a (-1) factor corresponding to $p_1, q_{\sigma(2)}$. Thus our overall term is

$$(-1)^{|p_1||q_2|} \langle q_1, p_1\rangle \langle q_2, p_2\rangle .$$

- $\boxed{(2,1):}$ We have $\sigma(2) = 1 < \sigma(1) = 2$ and $1 < 2$, so we pick up a copy of Q_{12}. We also have a (-1) factor from $\ell = 1$, $t = 2$:

$$Q_{12}(-1)^{|p_1||q_1|} \langle q_2, p_1\rangle \langle q_1, p_2\rangle .$$

$\boxed{(m_q, m_p) = (1,1):}$ We have one pair of the form $\langle q_i, q_j\rangle$ and one of the form $\langle p_i, p_j\rangle$. Since we require $i < j$, this must be $\langle q_1, q_2\rangle \langle p_1, p_2\rangle$. Since $\{1,2\} \setminus (I_t \cup J_t) = \varnothing$, there are no coefficients in this case. So we have

$$\langle q_1, q_2\rangle \langle p_1, p_2\rangle .$$

Example 3.4.15: Circle Product Wick's Theorem: $m = 1, n = 3$

$$\begin{aligned}\langle q_1, p_1 \circ p_2 \circ p_3 \rangle &= (-1)^{|p_2||p_3|} \langle p_1, p_2 \rangle \langle q_1, p_3 \rangle \\ &+ (-1)^{|p_2||p_3|} \langle p_1, p_3 \rangle \langle q_1, p_2 \rangle \\ &+ (-1)^{|p_3||p_1|} P_{12} \langle p_2, p_3 \rangle \langle q_1, p_1 \rangle .\end{aligned}$$

$$\begin{aligned}m_p &= \frac{1}{2}(n_p - n_q) + m_q \\ &= \frac{1}{2}(3-1) + m_q \\ &= 1 + m_q.\end{aligned}$$

$\boxed{(m_q, m_p) = (0,1):}$ This is the only possibility for m_q, m_p, but it breaks down into three subcases depending on how I_p, J_p are chosen. Each case will have a *single* $\langle q_1, p_k \rangle$, and thus no coefficients will be produced corresponding to the $\langle q, p \rangle$ pairs.

- $\boxed{I_p = \{p_1\}, J_p = \{p_2\}:}$ $b_p = 1$, $c_p = 2$, $a_p, d_p = 3$. We have $a \not< b$, so no P term appears.

$$(-1)^{|p_2||p_3|} \langle p_1, p_2 \rangle \langle q_1, p_3 \rangle .$$

- $\boxed{I_p = \{p_1\}, J_p = \{p_3\}:}$ $b_p = 1$, $c_p = 3$, $a_p, d_p = 2$. We have $a_p \not< b_p$, so our term is

$$(-1)^{|p_2||p_3|} \langle p_1, p_3 \rangle \langle q_1, p_2 \rangle .$$

- $\boxed{I_p = \{p_2\}, J_p = \{p_3\}\text{:}}$ $b_p = 2$, $c_p = 3$, $a_p, d_p = 1$. We have $a_p < b_p$, so our term is

$$(-1)^{|p_3||p_1|} P_{12} \langle p_2, p_3 \rangle \langle q_1, p_1 \rangle .$$

Since there is only one q term, there are no Q_{ij} factors that appear.

Example 3.4.16: Circle Wick's Theorem: Ungraded Ordinary Bialgebra

In the case of trivial grading and Q-structure there are no non-trivial coefficients, so Wick's theorem for the circle product reduces to the sum over all ways of pairing up all of the elements in $\{q_1, q_2, \ldots, p_n\}$ such that in each pairing $\langle x_i, x_j \rangle$, x_i appears before x_j in the sequence $q_1, q_2, \ldots, q_m, p_1, \ldots, p_n$.

$$\sum \sum \prod \langle q_i, q_j \rangle \langle p_i, p_j \rangle \langle q_i, p_j \rangle$$

There are a couple differences of note between Wick's theorem on the ordinary product and on the circle product:

1. $\boxed{\text{Types of Pairings:}}$ In contrast to our original version of Wick's theorem, which only has terms of the form $\langle q_i, p_j \rangle$, the circle product version includes terms like $\langle q_i, q_j \rangle$ and $\langle p_i, p_j \rangle$.

2. $\boxed{\text{Length:}}$ While $\langle \bar{q}, \bar{p} \rangle$ requires $|\bar{q}| = |\bar{p}|$ to be non-zero, $\langle \bar{q}^\circ, \bar{p}^\circ \rangle$ only needs the sum of the two lengths to be even.

Taken together, these features of Wick's theorem on the circle product almost allow us to conclude that this version of Wick's theorem is independent of which terms appear on which side of the pairing, since all possible pairing combinations appear. However, the $Q_{c,r}$ terms prevent this from happening, since it only depends on the terms on the left of the pairing. However, if $Q_{cr} = (-1)^{|q_c||q_r|}$, Wick's theorem only depends on the order of the terms and we arrive at the following:

Corollary 3.4.17

Let B be an ordinary Laplace graded bialgebra with primitive elements P. Then

$$\langle q_1 \circ \ldots \circ q_m, p_1 \circ \ldots \circ p_n \rangle = \langle q_1 \circ \ldots \circ q_m \circ p_1, p_2 \circ \ldots \circ p_n \rangle \quad (3.58)$$

for all $q_i, p_i \in P$.

Proof. This is a corollary of Proposition 3.4.13. $\square$

Correlation Functions: Because the pairing only depends on the order of the terms appearing in it in this case, we forget the fact that the pairing has a left and right side, and can reinterpret it as a collection of multilinear functionals, one on each B^n:

$$\langle p_1, \ldots, p_n \rangle = \varepsilon(p_1 \circ \cdots \circ p_n). \quad (3.59)$$

In terms of physics, we will think of the pairing on a circle product of n primitive elements as their ***n*-point correlation function**.

3.5 Examples of Laplace Bialgebras

Example 3.5.1: Laplace Bialgebra: Trivial Laplace Pairing

Let B be a bialgebra. Then the pairing defined by

$$\langle x, y\rangle = \varepsilon(xy) \tag{3.60}$$

is a Laplace pairing.

$\boxed{\text{Conditions:}}$ We have $\langle x, yz\rangle = \varepsilon(xyz) = \langle xy, z\rangle$. Now using counitality,

$$\begin{aligned}\sum \left\langle x_{(1)}, y\right\rangle \left\langle x_{(2)}, z\right\rangle &= \sum \varepsilon(x_{(1)}y)\varepsilon(x_{(2)}z) = \sum \varepsilon(x_{(1)})\varepsilon(y)\varepsilon(x_{(2)})\varepsilon(z)\\ &= \varepsilon(y)\varepsilon(z)\sum \varepsilon(x_{(1)})\varepsilon(x_{(2)})\\ &= \varepsilon(y)\varepsilon(z)\varepsilon(x).\end{aligned}$$

Similarly, $\sum \left\langle x, z_{(1)}\right\rangle \left\langle y, z_{(1)}\right\rangle = \varepsilon(x)\varepsilon(y)\sum \varepsilon(z_{(1)})\varepsilon(z_{(2)}) = \varepsilon(x)\varepsilon(y)\varepsilon(z)$. And $\langle 1, x\rangle = \langle x, 1\rangle = \varepsilon(x)$.

$\boxed{\text{Circle product:}}$ The circle product for the trivial pairing recovers the ordinary product in the Hopf algebra, since

$$\begin{aligned}x \circ y &= \sum \left\langle x_{(1)}, y_{(1)}\right\rangle x_{(2)}y_{(2)} = \sum \varepsilon(x_{(1)})\varepsilon(y_{(1)})x_{(2)}y_{(2)}\\ &= (\sum \varepsilon(x_{(1)})x_{(2)})(\sum \varepsilon(y_{(1)})y_{(2)})\\ &= xy.\end{aligned}$$

$\boxed{\text{Note:}}$ This is the type of pairing that we have in Frobenius algebras.

Thus we see that Laplace bialgebras will in general have a very different structure from their Frobenius counterparts; Frobenius-like pairings on Hopf algebras add nothing new from the perspective of the circle product.

Example 3.5.2: Laplace Pairing: Group Algebra

Laplace pairings on group algebras are bicharacters: group homomorphisms $G_{ab} \otimes G_{ab} \to k^\times$, where G_{ab} is the abelianization of G, such that

$$\begin{aligned} \langle xy, z\rangle &= \langle x, z\rangle \langle y, z\rangle \\ \langle x, yz\rangle &= \langle x, y\rangle \langle x, z\rangle \end{aligned}$$

for all $x, y, z \in G_{ab}$.

On a group Hopf algebra kG, the Laplace pairing condition is

$$\begin{aligned} \langle xy, z\rangle &= \langle x, z\rangle \langle y, z\rangle \\ \langle x, yz\rangle &= \langle x, y\rangle \langle x, z\rangle \end{aligned}$$

for all $x, y, z \in G$. It follows that

$$\langle xy, z\rangle = \langle x, z\rangle \langle y, z\rangle = \langle y, z\rangle \langle x, z\rangle = \langle yx, z\rangle .$$

Similarly,

$$\langle x, yz\rangle = \langle x, zy\rangle .$$

Thus $\langle x, - \rangle$ and $\langle -, x \rangle$ are trivial on commutators. Then by the universal property of the abelianization of G, G_{ab}, the maps $\langle x, - \rangle, \langle -, x \rangle : G \to k^\times$ correspond to morphisms $G_{ab} \to k^\times$. By uniqueness of the universal morphisms, it's easy to check that these maps are

$$\langle [x], - \rangle, \langle -, [x] \rangle : G_{ab} \to k^\times$$

where $[x]$ is the projection of x into the abelianization. By the universal property of the tensor product, it follows that these maps correspond to a unique bicharacter

$$G_{ab} \otimes G_{ab} \to k^\times.$$

$\boxed{\text{Terminology:}}$ The circle product induced from bicharacters is typically called the **twisted product** and kG under the circle product is called the **twisted group algebra**.

Example 3.5.3: Laplace Pairing: Universal Enveloping Algebra

Consider the universal enveloping algebra $U[\mathfrak{g}]$ of a Lie algebra $\mathfrak{g}$. A Laplace pairing on $U[\mathfrak{g}]$ is a bilinear map

$$\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \times \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to k.$$

A Laplace pairing on $U[\mathfrak{g}]$ takes the form

$$U[\mathfrak{g}] \otimes U[\mathfrak{g}] \to k.$$

The universal enveloping algebra is generated by all monomials in $\mathfrak{g}$.

Since each element of $\mathfrak{g}$ is primitive, we can apply Wick's theorem, Theorem 3.4.6, to determine the value of the Laplace pairing on any two elements, once we know its value on $\mathfrak{g}$.

$\boxed{\text{Laplace conditions:}}$ The Laplace pairing conditions applied to the Hopf algebra structure on $U[\mathfrak{g}]$ imply

$$\begin{aligned}\langle x, [y, z]\rangle &= \langle x, yz - zy\rangle = \langle x, yz\rangle - \langle x, zy\rangle \\ &= 0\end{aligned}$$

for all $x, y, z \in \mathfrak{g}$. Similarly $\langle [x, y], z\rangle = 0$. Thus any Laplace pairing induces a bilinear map

$$\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \times \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to k.$$

On the other hand, any such map extends to a pairing $\mathfrak{g} \times \mathfrak{g} \to k$ by $\langle x, y\rangle \equiv \langle \bar{x}, \bar{y}\rangle$, where $\bar{x}, \bar{y}$ are the projections into the quotient. This pairing then extends to unique Laplace pairing on $U[\mathfrak{g}]$ through Wick's theorem.

Definition 3.5.4: Twisted Group Algebra

Given a group Hopf algebra kG and Laplace pairing $\langle \cdot, \cdot\rangle$, the group algebra twisted by $\langle \cdot, \cdot\rangle$ is the set kG with product

$$g \circ h \equiv \langle g, h\rangle\, gh. \tag{3.61}$$

Example 3.5.5: Twisted Group Algebra: Non-Commutative Torus

Consider the group algebra on $\mathbb{C}\mathbb{Z}^2$, with the Laplace pairing

$$\langle (a,b),(c,d)\rangle = \exp(2\pi i\theta bc) \tag{3.62}$$

where θ is irrational. The induced twisted group algebra is called the **non-commutative torus**.

$\boxed{\text{Intuition:}}$ The non-commutative torus appears in non-commutative geometry as C^* algebra meant to model the continuous functions on a non-commutative space.

Chapter 4

Hopf-Frobenius Modules

In this chapter we combine the Frobenius and Hopf algebraic structure we've developed thus far, in the form of Hopf-Frobenius (HF) modules. These modules allow us to prove a version of the Lie correspondence from the perspective of Lie algebras and Lie groups as Hopf algebras (the universal enveloping algebra and group algebra, respectively) acting on Frobenius algebras.

4.1 Hopf-Frobenius Modules

Definition 4.1.1: Hopf-Frobenius Module

A **left Hopf-Frobenius module** is a pair (H, F) where H is a Hopf algebra, and F is a Frobenius algebra such that

1. F has the structure of a left H-module.
2. $\sum (-1)^{|x||g_{(2)}|} \left\langle g_{(1)}x, g_{(2)}y \right\rangle = \varepsilon_H(g) \left\langle x, y \right\rangle$.
3. $\left\langle gx, y \right\rangle = (-1)^{|g||x|} \left\langle x, S(g)y \right\rangle$

for all homogeneous $g \in H$ and $x, y \in F$.

$\boxed{\text{Convention:}}$ One can similarly define right Hopf-Frobenius modules. However, in what follows, we shall consider all of our Hopf-Frobenius modules to be *left* Hopf-Frobenius modules.

The third condition in the definition of Hopf-Frobenius modules will be useful in proving a version of the Lie group - Lie algebra correspondence for Hopf-Frobenius modules. However, for grouplike and primitive elements of the Hopf algebra, it follows from the previous conditions:

Lemma 4.1.2: •

Let H be a Hopf algebra and F be Frobenius algebras which is also a left H-module. If H and F satisfy

$$\sum (-1)^{|x||g_{(2)}|} \left\langle g_{(1)}x, g_{(2)}y \right\rangle = \varepsilon_H(g) \left\langle x, y \right\rangle$$

and g is grouplike or primitive, then $\left\langle S(g)x, y \right\rangle = (-1)^{|g||x|} \left\langle x, gy \right\rangle$.

Proof. $\boxed{\text{Primitive:}}$ Let p be primitive. The first condition for HF-modules becomes $\sum(-1)^{|x||p_{(2)}|}\left\langle p_{(1)}x, p_{(2)}y\right\rangle = \langle px, y\rangle + (-1)^{|x||p|}\langle x, py\rangle = \varepsilon(p)\langle x, y\rangle = 0$. So $\langle px, y\rangle = -(-1)^{|x||p|}\langle x, py\rangle = (-1)^{|x||p|}\langle x, (-p)y\rangle = (-1)^{|x||p|}\langle x, S(p)y\rangle$.

$\boxed{\text{Grouplike:}}$ Let g be grouplike. Then the first condition becomes $(-1)^{|x||g|}\langle gx, gy\rangle = \langle x, y\rangle$. Since g^{-1} is also grouplike, we have $\langle gx, y\rangle = (-1)^{|x||g^{-1}|}\langle g^{-1}gx, g^{-1}y\rangle = (-1)^{|x||g|}\langle x, S(g)y\rangle$. □

Lemma 4.1.3: ●

Let H be a Hopf algebra and F be a Frobenius algebra that is also a left H-module. If $g, h \in H$ satisfy

$$\sum(-1)^{|x||g_{(2)}|}\left\langle g_{(1)}x, g_{(2)}y\right\rangle = \varepsilon_H(g)\langle x, y\rangle$$

for all $x, y \in F$ then so does the product gh. Similarly, if g, h satisfy

$$\langle gx, y\rangle = (-1)^{|g||x|}\langle x, S(g)y\rangle$$

for all x, y, then so does gh.

Proof. $\boxed{\text{Condition 2:}}$ First note that

$$\begin{aligned}
\Delta(gh) &= \Delta(g)\Delta(h) \\
&= \left(\sum g_{(1)} \otimes g_{(2)}\right)\left(\sum h_{(1')} \otimes h_{(2')}\right) \\
&= \sum(-1)^{|g_{(2)}||h_{(1')}|} g_{(1)}h_{(1')} \otimes g_{(2)}h_{(2')}.
\end{aligned}$$

Using this, we can write the second Hopf-Frobenius condition in Definition 4.1.1 applied to the product gh as

$$\begin{aligned}
\sum(-1)^{|x||(gh)_{(2)}|}&\left\langle (gh)_{(1)}x,(gh)_{(2)}y\right\rangle \\
&=\sum(-1)^{|x||g_{(2)}h_{(2')}|+|g_{(2)}||h_{(1')}|}\left\langle (g_{(1)}h_{(1')})x,(g_{(2)}h_{(2')})y\right\rangle \\
&=\sum(-1)^{|x||g_{(2)}|+|x||h_{(2')}|+|g_{(2)}||h_{(1')}|}\left\langle g_{(1)}(h_{(1')}x),g_{(2)}(h_{(2')}y)\right\rangle \\
&=\sum(-1)^{|g_{(2)}||h_{(1')}x|+|x||h_{(2')}|}\left\langle g_{(1)}(h_{(1')}x),g_{(2)}(h_{(2')}y)\right\rangle \\
&=\varepsilon(g)\sum(-1)^{|x||h_{(2')}|}\left\langle h_{(1')}x,h_{(2')}y\right\rangle \\
&=\varepsilon(g)\varepsilon(h)\left\langle x,y\right\rangle \\
&=\varepsilon(gh)\left\langle x,y\right\rangle .
\end{aligned}$$

$\boxed{\text{Condition 3:}}$ Finally we check the third condition in Definition 4.1.1. We use the fact that

$$S(gh)=(-1)^{|g||h|}S(h)S(g).$$

Then we have

$$\begin{aligned}
\left\langle (gh)x,y\right\rangle &=\left\langle g(hx),y\right\rangle \\
&=(-1)^{|g||hx|}\left\langle hx,S(g)y\right\rangle \\
&=(-1)^{|g||h|+|g||x|+|h||x|}\left\langle x,S(h)S(g)y\right\rangle \\
&=(-1)^{|g||x|+|h||x|}\left\langle x,S(gh)y\right\rangle \\
&=(-1)^{|x||gh|}\left\langle x,S(gh)y\right\rangle .
\end{aligned}$$

□

4.2 Examples of Hopf-Frobenius Modules

Part of the motivation for Hopf-Frobenius modules is their ability to unify constructions that appear in geometry, algebraic topology, functional analysis, and algebra. We first introduce some of the main ways that Hopf-Frobenius modules can arise in nature, before investigating constructions that can be performed on these modules in more detail.

Example 4.2.1: HF-Module: Trivial Hopf Action

Let F be any Frobenius algebra and take our Hopf algebra to be the field k. Then there is a trivial Hopf algebra structure on (k, F) given by

$$\alpha \cdot f = \alpha f \tag{4.63}$$

for all $\alpha \in k$, $f \in F$.

Since $\Delta\alpha = \alpha 1 \otimes 1$, the HF conditions reduce to

$$\langle \alpha x, y \rangle = \alpha \langle x, y \rangle = \langle x, \alpha y \rangle .$$

4.2.1 Examples from Functional Analysis

Example 4.2.2: HF-Module: Schwartz Space with Derivatives

Let $F = \mathcal{S}(\mathbb{R}^n)$ be Schwartz space and $H = \mathbb{R}[\partial_1, \ldots, \partial_n]$ be a polynomial algebra whose action on F is given by partial derivatives. Then (H, F) is a Hopf-Frobenius module.

Since H is primitively generated, the Hopf-Frobenius conditions reduce to

$$\langle \partial_i f, g \rangle = - \langle f, \partial_i g \rangle \,, \tag{4.64}$$

or in integral notation, $\int (\partial_i f) g = - \int f \partial_i g$. This is precisely the integration by parts formula for functions (like those in F) which go to zero on the boundary.

$\boxed{\text{Similarly:}}$ $(k[\partial_1, \ldots, \partial_n], C_c^\infty(\mathbb{R}^n))$ is a Hopf-Frobenius module.

Example 4.2.3: HF-Module: Compactly Supported Functions with Derivations

Let M be an orientable smooth n-manifold equipped with a volume form $\omega \in \Omega^n(M)$. Take as Frobenius algebra $F = C_c^\infty(M)$ as in Example 2.3.11. Consider the Lie algebra of derivations $\mathrm{Der}(C_c^\infty(M))$, which is isomorphic to the Lie algebra of vector fields $\mathfrak{X}(M)$, which acts on $C_c^\infty(M)$. Let $H = U[\mathrm{Der}(C_c^\infty(M))]$ be the universal enveloping algebra on the derivations so that F is a left H-module. The pair (H, F) is a Hopf-Frobenius module.

Proposition 4.2.4

Let G be a locally compact Hausdorff topological group and consider the group Hopf algebra $H = kG$. Let $F = C_c(G)$ be the algebra of compactly supported functions on G, and give F the structure of a left H-module by $C_c(G)$ by $g \cdot \phi(h) = \phi(hg)$. Then given any measure μ on G such that the linear functional

$$\varepsilon_\mu(\phi) = \int \phi d\mu \tag{4.65}$$

has a simple kernel, (H, F) is a Hopf-Frobenius module if and only if μ is a Haar measure.

Proof. Since we're working with a Hopf algebra generated by grouplike elements, the Hopf-Frobenius conditions reduce to

$$\langle g\phi, g\psi \rangle = \langle \phi, \psi \rangle .$$

In integral notation, this is $\int \phi(xg)\psi(xg)d\mu(x) = \int \phi\psi$. Since we're working with compactly supported functions, given any such function, ϕ, and a $g \in G$, take a function 1^g_ϕ which is equal to one on at least the support of ϕ and the image of $\operatorname{supp}\phi$ under the action of g^{-1}, and goes to zero outside of them. Then

$$\int \phi(xg)d\mu(x) = \int \phi(xg)1^g_\phi(xg)d\mu(x) = \int \phi 1^g_\phi = \int \phi d\mu. \tag{4.66}$$

Thus ε is left-invariant with respect to G. This is equivalent to μ being a Haar measure on $C_c(G)$. Conversely, if μ is a Haar measure, it is obvious that the condition $\langle g\phi, g\psi \rangle = \langle \phi, \psi \rangle$ will be satisfied. $\square$

4.2.2 Examples from Geometry

Example 4.2.5: HF-Module: Lie Groups

Let $O(n)$ be the orthogonal group, and take $F = \mathbb{R}^n$ with the standard basis and let $H = \mathbb{R}O(n)$ be the group algebra of the orthogonal group. Letting $O(n)$ act on F by its standard representation and extending to H linearly, gives an HF-module.

We're grouplike-generated, so the HF-conditions become

$$\langle gx, gy \rangle = \langle x, y \rangle \, ,$$

which is just the condition that $O(n)$ acts orthogonally on $\mathbb{R}^n$.

$\boxed{\text{Similarly:}}$ We can construct HF-modules corresponding to most of the classical Lie groups: $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$, $O(m,n)$, $SO(m,n)$, $\mathrm{Spin}(n)$.

$\boxed{\text{Note:}}$ The above Lie groups fit very naturally into the Hopf-Frobenius framework because they preserve certain bilinear forms. In the next section, we will show how, for these types of form-preserving Lie groups, some of the main structure of Lie theory can be obtained simply by working within the Hopf-Frobenius world, without appealing to any explicit differential geometry.

Example 4.2.6: HF-Module: Lie Algebras

Let $\mathfrak{o}(n)$ be the orthogonal Lie algebra and take our Hopf algebra to be the universal enveloping algebra $U[\mathfrak{o}(n)]$. Let $F = \mathbb{R}^n$ with the standard basis. Let $\mathfrak{o}(n)$ act on F by its standard representation and extend to H by letting products of elements of $\mathfrak{o}(n)$ act on F by composition. This has the structure of an HF-module.

Being primitively generated, the HF-conditions are

$$\langle \phi x, y \rangle = - \langle x, \phi y \rangle .$$

for $\phi \in \mathfrak{o}(n)$. This says that elements of $\mathfrak{o}(n)$ are represented by skew-symmetric matrices, which is precisely what happens in the standard representation!

$\boxed{\text{Similarly:}}$ We can construct HF-modules corresponding to Lie algebras for the classical groups mentioned in the previous example.

4.3 The Lie Group-Lie Algebra Correspondence for Hopf-Frobenius Modules

In this section we recover the Lie group-Lie algebra correspondence for a certain class of Lie groups and algebras using the structure of Hopf-Frobenius modules. This allows us to motivate a definition of a Lie group and Lie algebra associated to a Hopf-Frobenius module.

Lemma 4.3.1

Let G be a group and consider the group algebra kG. The vector subspace of kG generated by

$$\{g - g^{-1} \mid g \in G\} \tag{4.67}$$

is a Lie algebra under the commutator bracket.

Proof. We check that the subspace is closed under the commutator. We have

$$\begin{aligned}
[g - g^{-1}, h - h^{-1}] &= (g - g^{-1})(h - h^{-1}) - (h - h^{-1})(g - g^{-1}) \\
&= gh - g^{-1}h - gh^{-1} + g^{-1}h^{-1} - (hg - h^{-1}g - hg^{-1} + h^{-1}g^{-1}) \\
&= (gh - h^{-1}g^{-1}) + (g^{-1}h^{-1} - hg) + (h^{-1}g - g^{-1}h) + (hg^{-1} - gh^{-1}).
\end{aligned}$$

Each of the four bracketed terms is of the form $x - x^{-1}$ with $x \in G$.

$\square$

This Lie algebra that we associate to a group has nothing to do with its topological structure — it really only sees G as a discrete group. It is quite surprising that this construction will yield the appropriate Lie algebra for all of the Lie groups we consider. Before we show this, however, we note that this Lie algebra construction can be generalized to apply to *any* Hopf algebra.

Lemma 4.3.2: Lie Algebra of a Hopf Algebra

Let H be an involutive Hopf algebra and consider the vector subspace generated by

$$\{x - S(x) \mid x \in H\}. \tag{4.68}$$

This is a Lie algebra under the commutator bracket.

Proof. This is essentially the same proof as Lemma 4.3.1, and follows from the fact that $S(xy) = S(y)S(x)$ and $S^2(x) = x$ for every element of a Hopf algebra.

$$\begin{aligned}
[x - S(x), y - S(y)] &= [x, y] - [x, S(y)] - [S(x), y] + [S(x), S(y)] \\
&= [x, y] - [x, S(y)] - S([S(y), x]) + S([y, x]) \\
&= [x, y] - S([x, y]) - ([x, S(y)] - S([x, S(y)]))\,.
\end{aligned}$$

Now the the first pair of terms and the second pair are both of the form $z - S(z)$. $\square$

$\boxed{\text{Note:}}$ In kG we have $S(g) = g^{-1}$, and since antipodes are linear, the linear combinations of all $\{x - S(x)\}$ will match the vector space generated by $\{g - g^{-1}\}$ used in the previous construction.

Definition 4.3.3: Lie Algebra of a Hopf Algebra

Let H be a Hopf algebra. The **Lie algebra associated to $\boldsymbol{H}$** is the vector space

$$\mathfrak{L}_H \equiv \{\sum_i \alpha_i(x - S(x)) \mid x \in H, S^2(x) = x, \alpha_i \in k\}, \tag{4.69}$$

with the commutator bracket.

When G is a group and H is the group Hopf algebra kG, we denote $\mathfrak{L}_H$ as $\mathfrak{L}_G$, if the field k is clear.

Proposition 4.3.4

Let $\mathfrak{L} : \text{Gr} \to \text{L.Alg}$ be the assignment from the category of groups to the category of Lie algebras which sends G to $\mathfrak{L}_G$ and $\phi : G \to H$ to the morphism $\mathfrak{L}_\phi$ defined by

$$\mathfrak{L}_\phi(g - g^{-1}) = \phi(g) - \phi(g)^{-1}. \tag{4.70}$$

This is a functor.

Proof. $\boxed{\text{Well-defined:}}$ Note that in kG the grouplike elements are linearly independent [12, Proposition 1.4.14]. It follows that the only linear dependence between different $g - g^{-1}$ is the relation $-(g - g^{-1}) = g^{-1} - (g^{-1})^{-1}$, which is respected by our definition of $\mathfrak{L}_\phi$. So $\mathfrak{L}_\phi$ is well-defined.

$\boxed{\text{Lie algebra map:}}$ We now check that $\mathfrak{L}_\phi$ is a Lie algebra morphism.

Using the fact that

$$[g - g^{-1}, h - h^{-1}] = (gh - h^{-1}g^{-1}) + (g^{-1}h^{-1} - hg) + (h^{-1}g - gh^{-1}) + (hg^{-1} - gh^{-1}),$$

we have

$$\begin{aligned}\mathfrak{L}_\phi([g - g^{-1}, h - h^{-1}]) &= \mathfrak{L}_\phi(gh - h^{-1}g^{-1}) + \ldots + \mathfrak{L}_\phi(hg^{-1} - gh^{-1}) \\ &= (\phi(gh) - \phi(gh)^{-1}) + \ldots + (\phi(hg^{-1}) - \phi(gh^{-1})^{-1}) \\ &= (\phi(g)\phi(h) - \phi(h)^{-1}\phi(g)^{-1}) + \ldots + (\phi(h)\phi(g^{-1}) - \phi(g)\phi(h)^{-1}) \\ &= [\phi(g) - \phi(g)^{-1}, \phi(h) - \phi(h)^{-1}].\end{aligned}$$

$\boxed{\text{Functorial:}}$ Let $G \xrightarrow{\phi} H \xrightarrow{\psi} K$. We have that $\mathfrak{L}_\psi\mathfrak{L}_\phi(g - g^{-1}) = \mathfrak{L}_\psi(\phi(g) - \phi(g)^{-1}) = \psi(\phi(g)) - \psi(\phi(g))^{-1} = \mathfrak{L}_{\psi\circ\phi}(g - g^{-1})$. $\square$

We next show that for Lie groups we're interested in, the above construction reproduces their standard Lie algebras.

Proposition 4.3.5

Let G be a Lie group over a field k that satisfies the following property:

- There exists a finite collection of non-degenerate bilinear forms on a vector space V such that $G = \{g \in \mathrm{End}(V) \mid \langle gx, gy\rangle_i = \langle x, y\rangle_i \text{ for all } i\}$, and the Lie algebra of G, $\mathfrak{g} = \{A \in \mathrm{End}(V) \mid \langle Ax, y\rangle_i = -\langle x, Ay\rangle_i \text{ for all } i\}$.

Then $\mathfrak{L}_G \cong \mathfrak{g}$.

Proof. Given an element of the Lie group, M, and a bilinear form $\langle \cdot, \cdot \rangle_i$ we have

$$\begin{aligned}\left\langle (M - M^{-1})x, y \right\rangle_i &= \langle Mx, y\rangle_i - \left\langle M^{-1}x, y \right\rangle_i \\ &= \left\langle x, M^{-1}y \right\rangle_i - \langle x, My \rangle_i \\ &= -\left\langle x, (M - M^{-1})y \right\rangle_i .\end{aligned}$$

So we have an assignment

$$\mathfrak{L} : \mathfrak{L}_G \to \mathfrak{g} \tag{4.71}$$

that sends $M - M^{-1} \mapsto M - M^{-1}$. This assignment induces a Lie algebra isomorphism $\mathfrak{L}_G \cong \mathfrak{g}$. $\square$

$\boxed{\text{Examples:}}$ The above class of Lie groups and Lie algebras covers a wide range, including most of the classical Lie groups/algebras: $O(n)$, $U(n)$, $Sp(2n)$, $O(p, q)$ (the indefinite orthogonal group), and in particular $\Lambda = O(1, 3)$ (the Lorentz group). Several additional Lie groups which don't satisfy the above condition nevertheless still reproduce the correct Lie algebra.

Proposition 4.3.6

Let G be $GL(n)$ or $SL(n)$ over a field k with $\mathrm{char}(k) \neq 2$, with corresponding Lie algebra $\mathfrak{g}$. Then

$$\mathfrak{L}_G \cong \mathfrak{g}. \tag{4.72}$$

Proof. $\boxed{GL(n):}$ Consider the invertible matrices $M = I+E_{ij} \in GL(n)$, $i \neq j$. We have $M - M^{-1} = 2E_{ij}$. Similarly, if $M = I + E_{ii} \in GL(n)$, then $M - M^{-1} = \frac{3}{2}E_{ii}$. Thus $\mathfrak{L}_G$ contains all of the generators of $\mathfrak{gl}(n)$, so $\mathfrak{L}_G \cong \mathfrak{gl}(n)$.

$\boxed{SL(n):}$ We again take $M = I+E_{ij} \in SL(n)$ to get the generators $2E_{ij}$. Our other generators are obtained from $M = I - \frac{1}{2}E_{ii} + E_{jj} \in SL(n)$. We have $M - M^{-1} = \frac{3}{2}(E_{jj} - E_{ii})$. This will produce all of the remaining generators (actually, only considering $M = I - \frac{1}{2}E_{11} + E_{jj}$ will be enough to produce the generators), and thus we have $\mathfrak{L}_G \cong \mathfrak{sl}(n)$. $\square$

Proposition 4.3.7: The Group Generated by Primitive Elements

Let H be a Hopf algebra. Let F be an algebra which is a left H-module, and let $\{\langle \cdot, \cdot \rangle_i\}_{i \in I}$ be finitely many bilinear forms which each give F the structure of a Frobenius algebra and turn (H, F) into a Hopf-Frobenius module (for the same fixed action by H). Let P be the primitive elements of H. Let $U[P]$ be the subalgebra generated by P and consider the induced action of $U[P]$ on F. Then

$$U[P]_F \equiv \{g \in U/\operatorname{ann} F \mid \langle gx, gy \rangle_i = \langle S(g)x, S(g)y \rangle_i = \langle x, y \rangle_i \text{ and } \langle S(g)x, y \rangle_i = \langle x, gy \rangle_i \text{ for all } i\}$$

forms a group, where $g^{-1} = S(g)$.

Proof. First notice that $g \in U[P]_F$ if and only if $S(g) \in U[P]_F$.

$\boxed{\text{Closed under Products:}}$ Given $g, h \in U[P]_F$, we have $\langle ghx, ghy\rangle = \langle hx, hy\rangle = \langle x, y\rangle$. Now we use the fact that the antipode is an antihomomorphism [12, Proposition 4.2.6] to write $\langle S(gh)x, S(gh)y\rangle = \langle S(h)S(g), S(h)S(g)y\rangle = \langle x, y\rangle$. Similarly, $\langle S(gh)x, y\rangle = \langle S(h)S(g)x, y\rangle = \langle S(g)x, hy\rangle = \langle x, ghy\rangle$. Thus $U[P]_F$ is closed under multiplication.

$\boxed{\text{Identity:}}$ Let 1 be the image of the unit of H in $U/\operatorname{ann} f$. It clearly satisfies the conditions to be in $U[P]_F$ and is the unit of the group.

$\boxed{\text{Inverses:}}$ Let $g \in U[P]_F$. By assumption $\langle x, y\rangle = \langle gx, gy\rangle = \langle S(g)gx, y\rangle$ for all y. Thus by non-degeneracy of the pairing, $(S(g)g-1)x = 0$ for all x. Since we've quotiented out the annihilator, this implies $S(g)g = 1$. Similarly, $\langle S(g)x, S(g)y\rangle = \langle x, gS(g)y\rangle$, giving $gS(g) = 1$. Thus

$$g^{-1} = S(g).$$

□

$\boxed{\text{Convention:}}$ We will denote Hopf-Frobenius modules of the form that appears in Proposition 4.3.7 by

$$(H, F, \{\langle\cdot,\cdot\rangle_i\}_{i\in I}).$$

In the case of only a single Frobenius algebra structure on F, this reduces to the ordinary notion.

Definition 4.3.8: Lie Group of a Hopf-Frobenius Module

Let $(H, F, \{\langle\cdot,\cdot\rangle_i\}_{i\in I})$ be as above and let U be the subalgebra generated by the primitive elements of H. We denote the group constructed above by

$$\mathcal{G}_F^H \equiv \{g \in U/\operatorname{ann} F \mid \langle gx, gy\rangle_i = \langle S(g)x, S(g)y\rangle = \langle x, y\rangle_i \\ \text{and } \langle S(g)x, y\rangle_i = \langle x, gy\rangle_i\,, \text{ for all } i\}$$

and call it the **Lie group of $(H, F, \{\langle\cdot,\cdot\rangle_i\}_{i\in I})$**.

A Lie group should be more than a discrete group. By [38, Corollary 3.41] we know that topological groups have at most one Lie group structure up to isomorphism. As such, we can aim to view the Lie group of a Hopf-Frobenius module just as a topological group. We do such in the next proposition.

Lemma 4.3.9

Let $(H, F, \{\langle\cdot,\cdot\rangle_i\}_{i\in I})$ be a Hopf-Frobenius module. Then $(k\mathcal{G}_F^H, F, \{\langle\cdot,\cdot\rangle_i\}_{i\in I})$ is also a Hopf-Frobenius module.

Proof. Since $U[P]$ acts on F and $\operatorname{ann} F$ is an ideal of $U[P]$, F is a left $\mathcal{G}_F^H$-module. The Hopf-Frobenius conditions are satisfied by definition, since for all $g \in \mathcal{G}_F^H$ and $i \in I$

$$\langle gx, gy\rangle_i = \langle x, y\rangle_i$$
$$\langle S(g)x, y\rangle_i = \langle x, gy\rangle_i\,. \qquad \square$$

Lemma 4.3.10

Let $(H, F, \{\langle\cdot,\cdot\rangle_i\}_{i\in I})$ be a Hopf-Frobenius module with Lie group $\mathcal{G}_F^H$, where F is finite dimensional and over a trivially-starred field, and is given the topology induced by all $\langle x, -\rangle_i, \langle -, y\rangle_i$. Give $\mathcal{G}_F^H$ the initial topology induced by the maps

$$\mathcal{G}_F^H \xrightarrow{-\cdot x} F \tag{4.73}$$

for all $x \in F$. Then the antipode map $S : \mathcal{G}_F^H \to \mathcal{G}_F^H$ is continuous.

Proof. $\boxed{\text{Antipode:}}$ From Proposition 4.3.7, the antipode map is the inverse map on $\mathcal{G}_F^H$, and in particular is involutive. Now since the topology on $\mathcal{G}_F^H$ is initial, we need to check that the compositions $\mathcal{G}_F^H \xrightarrow{S} \mathcal{G}_F^H \xrightarrow{(\cdot,x)} F$ are continuous. These are continuous if and only if the postcompositions with $\langle a, -\rangle_i$ and $\langle -, b\rangle_i$ are continuous for all $a, b \in F$. We have $\langle a, S(g)x\rangle_i = \langle S^2(g)a, x\rangle_i = \langle ga, x\rangle_i$ and $\langle S(g)x, b\rangle_i = \langle x, gb\rangle_i$. So we're looking at the following commutative diagram:

$$\begin{tikzcd}
 & F \arrow[r, "\langle\cdot,x\rangle_i"] & k \\
\mathcal{G}_F^H \arrow[ur, "(\cdot,a)"] \arrow[r, "S"] \arrow[dr, "(\cdot,x)"'] & \mathcal{G}_F^H \arrow[r, "(\cdot,x)"] & F \arrow[u, "\langle a,\cdot\rangle_i"'] \arrow[d, "\langle\cdot,b\rangle_i"] \\
 & F \arrow[r, "\langle x,\cdot\rangle_i"] & k
\end{tikzcd}$$

The outer maps are continuous by construction of the topologies on F and $\mathcal{G}_F^H$. Thus S is continuous. $\square$

Proposition 4.3.11: •

Let $(H, F, \{\langle\cdot,\cdot\rangle_i\}_{i\in I})$ be a Hopf-Frobenius module where F is finite-dimensional and over a trivially-starred field. Then the action of $\mathcal{G}_F^H$ on F

$$(\cdot,\cdot) : \mathcal{G}_F^H \times F \to F$$

is jointly continuous.

Proof. By the characteristic property of the initial topology, the action is continuous if and only if its post composition with each $\langle x, -\rangle_i$ and $\langle -, x\rangle_i$ is continuous. We have

$$\langle x, -\rangle_i \circ (\cdot,\cdot)[g, y] = \langle x, g \cdot y\rangle_i = \langle S(g) \cdot x, y\rangle_i \,.$$

Thus the following diagram commutes

$$\begin{tikzcd}
\mathcal{G}_F^H \times F \arrow[r, "S"] \arrow[d, "(\cdot,\cdot)"'] & \mathcal{G}_F^H \times F \arrow[r, "(\cdot,x)\times 1"] & F \times F \arrow[d, "\langle\cdot,\cdot\rangle_i"] \\
F \arrow[rr, "\langle x,-\rangle"'] & & F
\end{tikzcd}$$

By the definition of the topology on $\mathcal{G}_F^H$, Lemma 2.1.13, and Lemma 4.3.10 $(\cdot, x)$, $\langle\cdot,\cdot\rangle_i$, and S, respectively, are continuous, and thus $(\cdot,\cdot) \circ \langle x, -\rangle_i$ is continuous. Similarly, $\langle g \cdot y, x\rangle_i = \langle y, S(g) \cdot x\rangle_i$, so we have the commuting diagram

$$\begin{tikzcd}
\mathcal{G}_F^H \times F \arrow[r, "S"] \arrow[d, "(\cdot,\cdot)"'] & \mathcal{G}_F^H \times F \arrow[r, "(\cdot,x)\times 1"] & F \times F \arrow[r, "\tau"] & F \times F \arrow[d, "\langle\cdot,\cdot\rangle_i"] \\
F \arrow[rrr, "\langle -,x\rangle_i"'] & & & F
\end{tikzcd}$$

showing that $(\cdot,\cdot) \circ \langle -, x\rangle_i$. It follows that the action map is jointly continuous. $\square$

Proposition 4.3.12: The Lie Group of an HF-Module is a Topological Group

Let $(H, F, \{\langle\cdot,\cdot\rangle_i\}_{i\in I})$ be a Hopf-Frobenius module where F is finite-dimensional and over a trivially-starred field. The initial topology induced by the maps

$$\mathcal{G}_F^H \xrightarrow{-\cdot x} F \tag{4.74}$$

for all $x \in F$ gives $\mathcal{G}_F^H$ the structure of a topological group.

Proof. $\boxed{\text{Inverse:}}$ From Lemma 4.3.10, the inverse (i.e. antipode) is a continuous map.

$\boxed{\text{Multiplication:}}$ By the characteristic property of the weak topology, $\mu : \mathcal{G}_F^H \times \mathcal{G}_F^H \to \mathcal{G}_F^H$ is continuous if and only if $(\cdot, x) \circ \mu : \mathcal{G}_F^H \times \mathcal{G}_F^H \to F$ is continuous for all $x \in F$. Consider the diagram

$$\begin{tikzcd}
\mathcal{G}_F^H \times \mathcal{G}_F^H \arrow[r, "\mu"] \arrow[d, "1\times(\cdot,x)"'] & \mathcal{G}_F^H \arrow[d, "(\cdot,x)"] \\
\mathcal{G}_F^H \times F \arrow[r, "(\cdot,\cdot)"'] & F
\end{tikzcd}$$

This commutes due to the associativity of the Hopf-Frobenius action: $(g, h) \to (gh) \cdot x = g \cdot (h \cdot x)$. The left map of the above diagram is continuous by definition, and the bottom map is continuous from Proposition 4.3.11. It follows that the multiplication μ is continuous.

$\square$

Lemma 4.3.13

Let $\rho : \mathcal{A} \to \mathrm{End}(V)$ be a finite-dimensional faithful irreducible representation of a unital associative algebra over a field k such that there is a filtration of the form

$$\rho(A_m \dots A_1)(V) \subseteq \rho(A_{m-1} \dots A_1)(V) \subseteq \dots \subseteq \rho(A_1)(V) \subseteq V$$

where $A_i \in \mathcal{A}$ and $\dim \rho(A_m \dots A_1)(V) = 1$. Then

$$\mathcal{A} \cong \mathrm{End}(V). \tag{4.75}$$

Proof. $\boxed{\text{Transitivity:}}$ Note that since our representation is irreducible, the action of $\mathcal{A}$ is transitive on V, since given any non-zero $v \in V$, the subspace

$$\{Av \mid A \in \mathcal{A}\}$$

is invariant under the action of $\mathcal{A}$, and thus must be equal to all of V. Since we can produce a rank one matrix, $\rho(A_m \dots A_1)$, it follows that all rank one matrices can be obtained in the image of ρ. Since all matrices are in the span of rank one matrices, it follows that the map ρ is bijective, and thus $\mathcal{A} \cong \mathrm{End}(V)$. $\square$

Lemma 4.3.14

Let G be one of the following Lie groups with its standard representation:

$$\{O(n), SO(n) \mid n > 2\} \cup \{Sp(n)\}.$$

Then the induced representation of the universal enveloping algebra $U[\mathfrak{g}]$ surjects onto $\mathrm{End}(V)$ for the corresponding V.

Proof. For each case, by Lemma 4.3.13, it's sufficient to find an appropriate filtration by images of elements of $U[\mathfrak{g}]$. Note that each of the standard representations of $U[\mathfrak{g}]$ is irreducible. In each case that follows we fix some basis of our vector space and view the representations of $U[\mathfrak{g}]$ as matrix representations.

$\boxed{O(n):}$ For $n > 2$, the Lie algebra $\mathfrak{o}(n)$ in its standard representation has elements of the form $E_{ij} - E_{ji}$, and $E_{jk} - E_{kj}$, with i, j, k distinct. The product

$$(E_{ij} - E_{ji})(E_{jk} - E_{kj}) = -E_{jk}$$

is rank one. The same example holds for $\mathfrak{so}(n)$, since it's isomorphic to $\mathfrak{o}(n)$.

$\boxed{Sp(n):}$ The Lie algebra consists of block matrices of the form

$$\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix},$$

where B and C are symmetric matrices. In particular, we can take

$B = E_{ii}$ and every other block to be zero, to get a rank one matrix. $\square$

Lemma 4.3.15: •

Let $G = U(n)$ with $n > 2$ and consider the Hopf-Frobenius module given by $\mathbb{R}G$ acting on $\mathbb{R}^{2n}$. Then $U[\mathfrak{u}(n)]$ surjects onto $\mathrm{End}(\mathbb{R}^{2n})$.

Proof. The Lie algebra $\mathfrak{u}(n)$ consists of skew-Hermitian matrices. If $n > 2$, we can obtain all off-diagonal entries by using

$$(iE_{aa})(E_{ab} - E_{ba}) = iE_{ab}$$
$$(iE_{aa})(iE_{ab} + iE_{ba}) = -E_{ab}.$$

All diagonal entries are obtained from iE_{aa} and

$$(iE_{aa})^2 = -E_{aa}.$$ $\square$

Proposition 4.3.16

Let G be one of the Lie groups listed in Lemma 4.3.14 and let $\rho : G \to GL(F)$ be its standard representation on the corresponding vector space F with bilinear forms $\{\langle\cdot,\cdot\rangle_i\}_{i\in I}$. In particular, the action of G preserves the forms:

$$\langle gx, gy\rangle_i = \langle x, y\rangle_i\,, \text{ for all } g \in G, x, y \in F, i \in I. \tag{4.76}$$

If $\mathfrak{g}$ is the Lie algebra of G and $U[\mathfrak{g}]$ is its universal enveloping algebra then

$$\mathcal{G}_F^{U[\mathfrak{g}]} \cong G \tag{4.77}$$

as topological groups.

Proof. First note that in all of the cases in Lemma 4.3.14 our Frobenius algebra is finite-dimensional and over a trivially-starred field.

$\boxed{\text{Hopf-Frobenius module:}}$ The representation $\rho : G \to GL(F)$ induces a Lie algebra representation

$$d\rho : \mathfrak{g} \to \mathfrak{gl}(F). \tag{4.78}$$

It is from this representation that we get the structure of a Hopf-Frobenius module from the universal enveloping algebra of $\mathfrak{g}$ acting on F. Note that the primitive elements of the universal enveloping algebra $U[\mathfrak{g}]$ are precisely the elements of $\mathfrak{g}$, and these generate $U[\mathfrak{g}]$ as an algebra (taking all products and sums). So in this case $U[P]$, where P is the set of primitive elements of the Hopf algebra, which we use to construct $\mathcal{G}_F^{U[\mathfrak{g}]}$, is the universal enveloping algebra of $\mathfrak{g}$. From Lemma 4.3.14 we have a surjection

$$U[\mathfrak{g}] \to \mathrm{Mat}_n(k).$$

Since F is acted on by $\mathcal{G}_F^{U[\mathfrak{g}]}$, and each element of $\mathcal{G}_F^{U[\mathfrak{g}]}$ preserves the pairing, we have a group homomorphism $\phi : \mathcal{G}_F^{U[\mathfrak{g}]} \to G$. Now given any $g \in G$, we can view it as its matrix representation $\rho(g)$, which will correspond to some element in $\mathrm{Mat}_n(k)$ which preserves the pairing. It follows from the above surjection that there is some $\phi_g \in \mathcal{G}_F^{U[\mathfrak{g}]}$, which is mapped to g, and thus our group homomorphism is an isomorphism.

$\boxed{\text{Continuity:}}$ The standard representation of the group G is continuous, where the topology on matrices is the standard subspace topology induced from k^{n^2}. The topology on $\mathcal{G}_F^{U[\mathfrak{g}]}$ is homeomorphic to the subspace topology of the matrix topology. It follows that our bijection

$$G \to \mathcal{G}_F^{U[\mathfrak{g}]}$$

is an isomorphism of topological groups. $\square$

4.4 Exceptional Lie Groups

The above construction also works appropriately for the exceptional Lie groups. We present one small example below.

Example 4.4.1: Lie Group: F_4

The exceptional Lie group F_4 is the isometry group of the octonions. We can view the octonions as a non-associative Frobenius algebra over the real numbers. The above construction will yield

$$\mathcal{G}_{\mathbb{O}}^{U[\mathfrak{g}]} \cong F_4. \tag{4.79}$$

Chapter 5

Hopf-Frobenius Quantum Field Theory

5.1 Creation/Annihilation Operators

Definition 5.1.1: Bosonic Fock Space

Let V be a vector space of states (for example a Hilbert space), and let $\mathcal{S}[V]$ be the symmetric algebra on V. We give this algebra the structure of a comultiplication, defined on powers of elements of V by

$$\Delta f^n = \sum_k \sqrt{\frac{n!}{(n-k)!}} f^k \otimes f^{n-k} \tag{5.80}$$

and extended multiplicatively and linearly. We call $\mathcal{S}[V]$ with this comultiplication the **bosonic Fock space of** V.

$\boxed{\text{Fermions:}}$ Analogously, working with the exterior algebra and graded commutation rules allows us to define fermions and supersymmetric

Fock spaces.

$\boxed{\text{Intuition:}}$ f^n is interpreted as n bosons in the state f.

$$\Delta f^n = 1 \otimes f^n + \sqrt{n} f \otimes f^{n-1} + \sqrt{n(n-1)} f^2 \otimes f^{n-2} + \ldots + \sqrt{n!} f^n \otimes 1 \tag{5.81}$$

$\boxed{\text{Note:}}$ This coalgebra is *not co-commutative*. This is desirable, because it gives an interesting multiplicative structure on the convolution algebras over it. The structure is also not counital, however it is close — if we define, $\varepsilon(1) = 1$, and $\varepsilon(f^n) = 0$, we're counital up to a scalar:

$$\begin{aligned} \varepsilon_1(\Delta f^n) &= f^n \\ \varepsilon_2(\Delta f^n) &= \sqrt{n!} f^n. \end{aligned} \tag{5.82}$$

The coalgebra is *not coassociative* either:

$$\Delta_1 \Delta f^n = \sum_{k\ell} \sqrt{\frac{n!}{(n-k)!}} \sqrt{\frac{k!}{(k-\ell)!}} f^\ell \otimes f^{k-\ell} \otimes f^{n-k},$$

while

$$\begin{aligned} \Delta_2 \Delta f^n &= \sum_{km} \sqrt{\frac{n!}{(n-k)!}} \sqrt{\frac{(n-k)!}{(n-k-m)!}} f^k \otimes f^m \otimes f^{n-k-m} \\ \Delta_2 \Delta f^n &= \sum_{km} \sqrt{\frac{n!}{(n-k-m)!}} f^k \otimes f^m \otimes f^{n-k-m}. \end{aligned}$$

Consider the coefficient of $f \otimes f \otimes f^{n-2}$ in each case:

$$\Delta_1 \Delta f^n = \sqrt{\frac{n!}{(n-2)!}} \sqrt{\frac{2!}{(1)!}} f \otimes f \otimes f^{n-2} + \dots$$
$$\Delta_2 \Delta f^n = \sqrt{\frac{n!}{(n-2)!}} f \otimes f \otimes f^{n-2} + \dots$$

We can view bosonic Fock space as a *generalized* bialgebra.

Definition 5.1.2: Creation/Annihilation Operators

Let C be a coalgebra with basis $\{x_i\}$. If $\Delta x_i = \sum (i \mid j,k) x_j \otimes x_k$, then the **creation and annihilation operators** $\boldsymbol{a_i, a_i^\dagger}$ with respect to the basis $\{x_i\}$ are endomorphisms on C defined by

$$\begin{aligned} a_j^\dagger x_k &= \sum_i (i \mid j,k) x_i \\ a_j x_k &= \sum_i (k \mid j,i) x_i. \end{aligned} \tag{5.83}$$

Joni and Rota presented this notion of creation/annihilation operators in [19].

Example 5.1.3: Creation/Annihilation Operators: Bosonic Fock Space

Applying Joni-Rota's construction to $\mathcal{S}[A]$, we find that $a_f f^n$ will be the term in the expansion of Δf^n of the form $f \otimes f^{n-1}$. Similarly, $a_f^\dagger f^n$ will be the term of the form $f \otimes f^n$ in Δf^{n+1}. Thus

$$\begin{aligned} a_f^\dagger f^n &= \sqrt{n+1} f^{n+1} \\ a_f f^n &= \sqrt{n} f^{n-1}. \end{aligned} \tag{5.84}$$

This is precisely the behaviour of the ordinary creation/annihilation operators from quantum theory.

5.2 Hopf Quantization

The circle product of a Laplace Hopf algebra can be interpreted as a method of quantizing the observables of the original Hopf algebra. We outline the general motivation and procedure for Hopf-quantization below, before moving on to specific examples.

$\boxed{\text{States/Operators:}}$ Our set-up will involve a Frobenius algebra of classical states F, and a Hopf algebra of classical operators, H_c, which form a Hopf-Frobenius module. The quantization procedure will then produce a new quantized Hopf-Laplace algebra H_Q, which has both an ordinary multiplication and a circle product.

$\boxed{\text{Two multiplications:}}$ The fact that we will end up with a structure that has two different multiplications (only one of which is compatible with the comultiplication in general) might seem odd, however it will turn out to naturally describe the two different roles that operators play: as composable maps (the ordinary multiplication) and as an algebra that have their own internal multiplication (the circle product).

Example 5.2.1: Deformation Quantization

Let M be a symplectic manifold and let $\mathfrak{p} = C^\infty(M)$ be the Poisson algebra of functions over the manifold. If we take our Hopf algebra to be $H_c = T[\mathfrak{p}]$, the tensor algebra of the underlying Lie algebra of $\mathfrak{p}$, as an ungraded Hopf algebra, there is a quantized Hopf algebra which corresponds to deformation quantization of the Poisson algebra $\mathfrak{p}$.

$\boxed{\text{Motivation:}}$ In classical mechanics, classical observables are viewed as functions on a symplectic manifold called phase space. These functions form a Poisson algebra, which we intend to quantize into a space of quantum observables.

$\boxed{\text{Laplace pairing:}}$ We want to interpret our pairing as an expectation value with respect to a fixed state. Recall that the Laplace pairing satisfies

$$\langle \phi, 1 \rangle = \varepsilon(\phi).$$

For all primitive ϕ, we have $\varepsilon(\phi) = 0$, and thus in our interpretation, the expectation of each classical observable is just zero. For this to make sense, we interpret ε as the expectation value with respect to a *zero state* (one with zero energy, zero momentum, located at the origin, and so on). We next modify our counit to take the expectation value with respect to other states.

$\boxed{\text{Modified counits:}}$ Given an element of phase space, $f \in M$, we define the modified counit

$$\varepsilon_f(\phi) \equiv \phi(f)$$

for all $\phi \in \mathfrak{p}$. Intuitively, for classical states without uncertainty, the expectation value of an observable on a state should be the actual value of that observed quantity. In order to remain counital, we define new comultiplications

$$\Delta_f \phi = 1 \otimes \phi + \phi \otimes 1 - \phi(f) 1 \otimes 1 \tag{5.85}$$

which we extend multiplicatively.

$\boxed{\text{Pairing:}}$ We now define a pairing that corresponds to the classical Hopf algebra:

$$\langle \phi, \psi \rangle_f \equiv \phi(f)\psi(f). \tag{5.86}$$

$\boxed{\text{Circle product:}}$ The circle product induced by the above pairing gives

$$\begin{aligned}
\phi \circ_f \psi = {} & \langle 1, 1 \rangle \phi\psi + \langle 1, \psi \rangle \phi - \psi(f) \langle 1, 1 \rangle \phi + \langle \phi, 1 \rangle \psi + \langle \phi, \psi \rangle 1 \\
& - \psi(f) \langle \phi, 1 \rangle 1 - \phi(f) \langle 1, 1 \rangle \psi - \phi(f) \langle 1, \psi \rangle 1 + \phi(f)\psi(f) \langle 1, 1 \rangle 1 \\
= {} & \phi\psi + \psi(f)\phi - \psi(f)\phi + \phi(f)\psi + \langle \phi, \psi \rangle 1 - \psi(f)\phi(f)1 \\
& - \phi(f)\psi - \phi(f)\psi(f)1 + \phi(f)\psi(f)1.
\end{aligned}$$

Cancelling out the matching terms, we arrive at

$$\phi \circ_f \psi = \phi\psi + (\langle \phi, \psi \rangle_f - \phi(f)\psi(f))1. \tag{5.87}$$

Thus with the choice $\langle \phi, \psi \rangle = \phi(f)\psi(f)$, we recover the ordinary product in the tensor algebra. By post-composing this with the map induced by the product $\mathfrak{p} \otimes \mathfrak{p} \to \mathfrak{p}$, we can interpret the choice of pairing $\langle \phi, \psi \rangle = \phi(f)\psi(f)$ as encoding the *classical* version of $\mathfrak{p}$.

$\boxed{\text{Quantizing operators:}}$ A quantization of our classical Laplace algebra now corresponds to picking a different Laplace pairing.

$\boxed{\text{Induced commutator:}}$ If we define $[\phi, \psi]_f^{\langle \cdot, \cdot \rangle} \equiv \phi \circ_f \psi - \psi \circ_f \phi$, we have

$$[\phi, \psi]_f^{\langle \cdot, \cdot \rangle} = [\phi, \psi] + (\langle \phi, \psi \rangle_f - \langle \psi, \phi \rangle_f)1. \tag{5.88}$$

We can interpret this as the **Moyal bracket** of deformation quantization, if we take $(\langle \phi, \psi \rangle_f - \langle \psi, \phi \rangle_f)$ to be a formal power series $\sum_{k=2}^{\infty} \alpha_k \hbar^k$ (either by changing our definition so that we map into $k[[\hbar]]$ instead of k, or just fixing a choice of real number $\hbar$). In this way we can recover the ordinary version of deformation quantization in the language of Hopf algebras.

Example 5.2.2: Quantization: Bosonic Fock Space

Let $\mathcal{S}[A]$ be the bosonic Fock space for the algebra A and let H be the tensor algebra generated by the creation and annihilation operators $\{a_f, a_f^\dagger\}$. There is a quantization procedure which yields the **normal ordering** on creation/annihilation operators.

We give the creation and annihilation operators primitive comultiplication

$$\begin{aligned} \Delta a_f^\dagger &= 1 \otimes a_f^\dagger + a_f^\dagger \otimes 1 \\ \Delta a_f &= 1 \otimes a_f + a_f \otimes 1 \end{aligned} \tag{5.89}$$

and extend multiplicatively. Let $\langle \cdot, \cdot \rangle$ be our Laplace pairing (to be specified momentarily).

$\boxed{\text{Circle product:}}$ The circle product

$$\phi \circ \psi \equiv \sum \left\langle \phi_{(1)}, \psi_{(1)} \right\rangle \phi_{(2)} \psi_{(2)} \tag{5.90}$$

on our bialgebra gives

$$\begin{aligned} a_f \circ a_g^\dagger &= 1 \left\langle a_f, a_g^\dagger \right\rangle 1 + 1 \left\langle a_f, 1 \right\rangle a_g^\dagger + a_f \left\langle 1, a_g^\dagger \right\rangle 1 + a_f \left\langle 1, 1 \right\rangle a_g^\dagger \\ &= \left\langle a_f, a_g^\dagger \right\rangle 1 + a_f a_g^\dagger. \end{aligned}$$

Similarly,

$$a_f^\dagger \circ a_g = \left\langle a_f^\dagger, a_g \right\rangle 1 + a_f^\dagger a_g.$$

If we define our pairing by

$$\begin{aligned} \left\langle a_f, a_g^\dagger \right\rangle &= \delta_{f,g} \\ \left\langle a_f^\dagger, a_g \right\rangle &= 0 \end{aligned} \tag{5.91}$$

then following Fauser [13], we can prove that the circle product gives the **normal ordering** on all products of creation/annihilation operators.

$\boxed{\text{Note:}}$ The pairings $\left\langle a_f^\dagger, a_g \right\rangle = 0$ and $\langle a_f, a_g^\dagger \rangle = \delta_{f,g}$ are precisely the values of the correlation functions of creation/annihilation operators in quantum mechanics and field theory. We interpret these functions as vacuum expectation values of the operators.

$\boxed{\text{Time-ordering:}}$ Time-ordered products are another important product of creation/annihilation operators employed by physicists. By definition, the time-ordered product is

$$x \circ_T y \equiv xy - x \circ y,$$

since $x \circ y$ is the normal-ordered product.

5.3 Hopf-Frobenius Field Theory

In this section we present our definition of a Hopf-Frobenius field theory. This notion is intended to cover both classical and quantum systems, with appropriate choices of Hopf and Frobenius algebras. The main idea behind our definition will be to have *three different* Hopf-Frobenius modules, which encode the spacetime geometry, local states or fields of the system, and the internal structure of the states and fields (such as spin). The way these three different HF-modules interact, through morphisms between them, will relate to physically interesting phenomena.

$\boxed{\text{Presentation:}}$ To aid in understanding, before presenting the full definition, we will look at how Hopf-Frobenius modules capture the relevant geometric, analytic, and algebraic information individually, before describing how they interact.

$\boxed{\text{Background:}}$ For more information on the concepts from quantum field theory which we investigate in this chapter, see [29, Part I] and [34, Part I].

5.3.1 Spacetime Geometry

The set-up of a quantum or classical field theory involves a spacetime manifold M, which carries a pseudo-Riemannian metric. This means that locally the tangent space looks like $\mathbb{R}^n$ with a non-degenerate bilinear form. Choosing a particular point of our manifold, we let our Frobenius algebra be the tangent space at that point: $F_{ST} = \mathbb{R}^n$ with the bilinear form given by the metric.

$\boxed{\text{Terminology:}}$ The elements of F_{ST} are often called **4-vectors** in physics.

$\boxed{\text{Symmetries:}}$ The isometries of the metric on spacetime are the **Poincaré transformations**. The subset of linear transformations are the **Lorentz transformations** $\Lambda(M)$. We can choose to take as our Hopf algebra of spacetime symmetries either the Lorentz group or the corresponding Lorentz Lie algebra, since we're working locally. We take

$$H^G_{ST} = k\Lambda$$
$$H^{\mathfrak{g}}_{ST} = U[\mathfrak{l}],$$

where $\mathfrak{l}$ is the Lorentz Lie algebra, as our Hopf algebras of spacetime symmetries. Both (H^G_{ST}, F_{ST}) and $(H^{\mathfrak{g}}_{ST}, F_{ST})$ form Hopf-Frobenius modules.

5.3.2 Fields

Fields of a classical field theory are sections of a bundle over our spacetime manifold. On the other hand, quantum fields are meant to be operator-valued distributions on the spacetime manifold.

$\boxed{\text{Local functions:}}$ Since our spacetime manifold looks locally like $\mathbb{R}^n$, we take as our Frobenius algebra of local functions F_f, the compactly supported functions $C_c(\mathbb{R}^n)$ or the Schwartz functions $\mathcal{S}(\mathbb{R}^n)$. These functions will not generally be the fields themselves. Instead, they will serve as local information that we can attach our fields to; in this way, they serve a role similar to test functions.

$\boxed{\text{Derivatives:}}$ A key tool in physics calculations is differential calculus. We attach this into our field theory by taking the Lie algebra of derivations on our Frobenius algebra of local functions, $\mathrm{Der}(F_f)$, and then taking its universal enveloping algebra, to get a Hopf algebra of local functions:

$$H_f = U[\mathrm{Der}(F_f)]. \tag{5.92}$$

Together, (H_f, F_f) form a Hopf-Frobenius module, where the action of the primitive elements of H_f is just the natural action of the derivations, and the action of products is the composition of derivations.

5.3.3 Internal Symmetries

Thus far, we've described the basic elements of what might be called a 'bare field theory' — a field theory without spin, gauge symmetry, or states in a Hilbert space. In order to introduce these, we create a Hopf-Frobenius module of what we call *internal structure*.

$\boxed{\text{Hilbert space:}}$ States of a quantum system are normally described by a Hilbert space. As we've seen in Example 2.3.10, Hilbert spaces can be viewed as Frobenius algebras once we choose a basis. This choice of basis is essential from a physics perspective — it corresponds to the fact that in quantum theory we need to be aware of the order that we measure observables in.

From our perspective, the choice of basis of our Hilbert space will reflect the current observable we are trying to measure. For commuting observables, there exists a simultaneous eigenbasis that we can choose, at which point the specific basis we're using can be ignored. The key point, however, is that from the Frobenius algebra perspective, a choice of basis for our Hilbert space is not optional: it comes from the requirement that we have an *algebra*, rather than simply a vector space, with a bilinear form. In this sense, the potential quantum nature of states is present from the beginning in the Hopf-Frobenius set-up.

$\boxed{\text{States:}}$ There is nothing a priori that forces us to choose our space of states to be a Hilbert space. In general, rather than a space of states, we would like to think about the relevant Frobenius algebra as one of *internal structure* — this may be a state structure for things like energy, or spin, or colour charge, or it could be something more general. For now we will think of our internal structure space F_I as some Hilbert space with a fixed basis.

$\boxed{\text{Symmetries:}}$ The symmetries of our internal structure will encode things like gauge symmetry and symmetry of states under global phase changes. Our central example that we will keep in mind will be

$$H_I = kG_I \tag{5.93}$$

where G_I is some Lie group of internal symmetries. However, just as in the case of spacetime symmetries, it will be useful to also have a Lie algebra version of internal operations

$$H_I^{\mathfrak{g}} = U[\mathfrak{b}]$$

where $\mathfrak{b}$ is some Lie algebra of skew-Hermitian operators.

With all of our basic structure in place, we can begin to describe how morphisms between the three Hopf-Frobenius modules give rise to physical phenomena. We first consolidate our structure into a very loose definition of a field theory (to which we will add more structure later).

Definition 5.3.1: Hopf-Frobenius Quantum Field Theory

A **Hopf-Frobenius quantum field theory** is a collection of three Hopf-Frobenius modules, (H_f, F_f), (H_{ST}, F_{ST}), and (H_I, F_I) which are meant to model the local functions, spacetime, and internal structure of the theory, as discussed above.

5.3.4 Fields and Particles

In quantum field theory the structure of fields (even classical ones) can be very subtle. A field φ may be a scalar field, vector field, a kind of spinor field (Dirac, Weyl, or Majorana), bosonic, fermionic, a singlet, doublet or triplet, right or left handed, and on and on.[1]

Below we try to present an approach to understanding how to distinguish different kinds of fields in terms of the Hopf-Frobenius formalism, which will hopefully be easily readable by researchers both in and outside of quantum physics.

[1]In fact some of these categories overlap with each other! We leave the question of which as an exercise for the reader.

Definition 5.3.2: Fields

Given three Frobenius algebras (F_f, F_{ST}, F_I) encoding local fields, spacetime, and internal structure respectively, a **field** of the corresponding field theory is an element of the tensor product

$$\varphi(x) = \sum_{ijk} \varphi_i(x) \otimes e_j \otimes \theta_k \in F_f \otimes F_{ST} \otimes F_I. \tag{5.94}$$

$\boxed{F_f \otimes F_{ST}:}$ In physics literature, the elements of $F_f \otimes F_{ST}$ are often collected together and labelled with an index. So we might see a field written as φ_μ, where $0 \leq \mu \leq 3$. What this would refer to is a field which has four components. You could express this field as $(\varphi_0(x), \ldots, \varphi_3(x))$, or, in our notation,

$$\varphi_0(x) \otimes e_0 + \ldots + \varphi_3(x) \otimes e_3.$$

Definition 5.3.3: Particle Type of a Field

Given Frobenius algebras (F_f, F_{ST}, F_I) and corresponding Hopf-algebras (H_f, H_{ST}, H_I), the choice of Hopf-Frobenius module structure on (H_{ST}, F_{ST}) will be called the **particle type** of the fields in the theory.

$\boxed{\text{Particles:}}$ In physics, representations of the Lorentz group correspond to different types of particles. Classic work in the area tells us that the representations are indexed by numbers which we interpret as the spin of the particles. The representation of the Lorentz group will be given by the choice of Hopf-Frobenius module (H_{ST}, F_{ST}).

Example 5.3.4: Particle Type: Scalar Fields

Scalar fields are spin 0 fields, and take the form

$$\phi : \mathbb{R}^4 \to k.$$

where k is a field, normally taken to be $\mathbb{C}$. This corresponds to the trivial representation of the Lorentz group.

In physics we normally think of a scalar field as just taking a single value, $\phi(x) \in \mathbb{C}$. However, to make things clearer, we will think of a scalar field as an element of $F_f \otimes F_{ST}$, where F_{ST} is four-dimensional, but all of the components of are forced to be the same. In other words, our field has the form

$$\phi = \sum_i \phi \otimes e_i = \phi \otimes (\sum_i e_i). \tag{5.95}$$

We can think of replacing F_{ST} with the one-dimensional Lorentz-invariant subspace generated by $\sum_i e_i$, say V. Then $F_f \otimes V \cong F_f$, so scalar fields appear as ordinary scalar-valued functions.

Example 5.3.5: Particle Type: Weyl Spinors

Weyl spinors have spin $1/2$ and come in two flavours: left-handed Weyl spinors come from the representation $(1/2, 0)$ of the Lorentz group, while right-handed Weyl spinors correspond to the $(0, 1/2)$ representation.

For each kind of spinor, we can think of restricting to a two-dimensional subspace of F_{ST}, or simply requiring the fields $F_f \otimes F_{ST}$ to satisfy certain restrictions on their components.

5.3.5 Connections and Derivatives

$\boxed{\text{Fields:}}$ The actual fields of our theory will be elements of the tensor product

$$F_f \otimes F_I.$$

Consider the case of a real vector bundle B over our spacetime manifold M. Picking a local trivialization, we can think of sections of the bundle as maps $\mathbb{R}^n \to \mathbb{R}^m$. Taking $F_f = C_c(\mathbb{R}^n)$ and $F_I = \mathbb{R}^m$, elements of the tensor product are of the form

$$f(\bar{x}) = \sum_{i=1}^{m} f_i(\bar{x}) \otimes e_i. \tag{5.96}$$

We view $f_i(\bar{x}) \otimes e_i$ as the component of the function f in the direction e_i.

Definition 5.3.6: Exterior Derivative

Let $f \in F_f$ be a local function with partial derivative actions $\partial_j \in H_f$. The **exterior derivative** of f is the map $d : F_f \to F_f \otimes F_I$ defined by

$$df_i = \sum_j \partial_j f_i \otimes e_j. \tag{5.97}$$

> Our notion of exterior derivative transforms local functions into fields (sections of a bundle). If we think of the example of the cotangent bundle, this is the ordinary behaviour of the exterior derivative: it sends smooth functions to covector fields.

Spacetime Connection: In Riemannian geometry, curvature of a manifold can be described by the data of a connection. We introduce the following notion of a connection on a Frobenius algebra, which we can use to describe both the Christoffel symbols for spacetime, and the gauge fields of gauge theories.

Definition 5.3.7: Connection on a Frobenius Algebra

Let F_f be a fixed Frobenius algebra. Given a Frobenius algebra F_2, a $\boldsymbol{F_f}$**-connection on** $\boldsymbol{F_2}$ is a choice of second Frobenius algebra F_1 and linear morphism

$$\nabla : F_2 \to F_f \otimes F_1 \otimes F_2. \tag{5.98}$$

When F_f and F_1 are clear, we will simply refer to ∇ as the connection.

Definition 5.3.8: Covariant Derivative

Let $\nabla : F_2 \to F_f \otimes F_1 \otimes F_2$ be a connection, and $d : F_f \to F_f \otimes F_1$ be an exterior derivative. The **covariant derivative with respect to** $\boldsymbol{\nabla}$ is the map $D_\nabla : F_f \otimes F_2 \to F_f \otimes F_1 \otimes F_2$ defined by

$$D_\nabla(\sum_i f_i(x) \otimes \theta_i) = \sum_i df_i(x) \otimes \theta_i + f_i(x)(\nabla\theta_i)_{(1)} \otimes (\nabla\theta_i)_{(2)} \otimes (\nabla\theta_i)_{(3)}.$$

Note the use of Sweedler notation in the above equation. If we pick a basis $\{e_i\}$ for F_1 and $\{\theta_j\}$ for F_2, we can express

$$\nabla\theta_j = \sum_{jk} A^k_{ij}(x) \otimes e_i \otimes \theta_k.$$

The covariant derivative then takes the form

$$D_\nabla(\sum_i f_i(x) \otimes \theta_i) = \sum \left(\partial_j f_i(x) \otimes e_j \otimes \theta_i + f_i(x) A^k_{ij}(x) \otimes e_i \otimes \theta_k\right).$$

It may not be immediately clear how this definition of 'connection' is related to the one from differential geometry. The following two examples show how connections from general relativity and gauge theory can be expressed in the above language.

Directional derivative: We define the **directional derivative** ∇_v for $v \in F_1$ to be the map

$$\nabla_v : F_f \otimes F_2 \to F_f \otimes F_2$$

defined by

$$\nabla_v(f \otimes \theta) \equiv \langle v, D_\nabla(f \otimes \theta)\rangle_2\,, \tag{5.99}$$

where $\langle\cdot,\cdot\rangle_2$ denotes pairing v with the second tensor factor in each term in $D_\nabla(f \otimes \theta)$.

Curvature: Let $\{e_i\}_{i\in I}$ be a basis for F_1. The **curvature of a connection $\boldsymbol{\nabla}$** with respect to the basis $\{e_i\}_{i\in I}$ is the map

$$[\nabla_i, \nabla_j] : F_f \otimes F_2 \to F_f \otimes F_2 \tag{5.100}$$

defined by

$$[\nabla_i, \nabla_j](f \otimes \theta) \equiv \nabla_i(f \otimes \theta)\nabla_j(f \otimes \theta) - \nabla_j(f \otimes \theta)\nabla_i(f \otimes \theta),$$

where ∇_k is the directional derivative with respect to e_k.

Example 5.3.9: Connection on Spacetime

A **connection on spacetime** will be a connection of the form

$$\nabla : F_{ST} \to F_f \otimes F_{ST} \otimes F_{ST}.$$

Say that our spacetime Frobenius algebra is $\mathbb{R}^4$ with a chosen basis $\{e_0, \ldots, e_3\}$, where e_0 corresponds to the time direction. Expanding the connection in the basis, we have

$$\nabla e_j = \sum_{ik} \Gamma^k_{ij}(x) \otimes e_i \otimes e_k. \tag{5.101}$$

The functions $\Gamma^k_{ij}(x)$ are called the **Christoffel symbols**.

$\boxed{\text{Covariant derivative:}}$ Our covariant derivative will take the form

$$\begin{aligned} D_\nabla f &= df + \sum_i f_i \otimes \nabla e_i \\ &= \sum_{ij} \partial_j f_i \otimes e_j \otimes e_i + \sum_{ijk} f_i(x)\Gamma^k_{ji}(x) \otimes e_j \otimes e_k. \end{aligned}$$

The directional derivative will be

$$\langle v, D_\nabla f\rangle_2 = \sum_{ij} v_j \partial_j f_i \otimes e_i + \sum_{ijk} \Gamma^k_{ji} v_j f_i \otimes e_k, \tag{5.102}$$

which matches with the usual form of the covariant directional derivative in differential geometry, in terms of the Christoffel symbols.

Example 5.3.10: Gauge Connection

A **gauge connection** is a connection of the form

$$\nabla : F_I \to F_f \otimes F_{ST} \otimes F_I.$$

Let $\{e_i\}$ be a basis for F_{ST} and $\{\theta_j\}$ be a basis for F_I.

Gauge Potential: If we expand out in a basis, we can write

$$\nabla \theta_j = \sum_{ik} A_{ij}^k(x) \otimes e_i \otimes \theta_k. \tag{5.103}$$

The terms $A_{ij}^k(x)$ are referred to in physics as the **gauge field** of the gauge connection[a].

Covariant derivative: Our covariant derivative will take the form

$$D_\nabla f = \sum_{ij} \partial_j f_i \otimes e_j \otimes \theta_i + \sum_{ijk} f_i(x) A_{ij}^k(x) \otimes e_j \otimes \theta_k.$$

Directional derivative:

$$\langle v, D_\nabla f\rangle_2 = \sum_{ij} v_j \partial_j f_i \otimes \theta_i + \sum_{ijk} v_j f_i(x) A_{ij}^k(x) \otimes \theta_k.$$

[a]Strictly speaking, the gauge field is normally taken to be $igA_{ab}^c(x)$, where g is called the *coupling parameter*.

5.4 Examples

5.4.1 U(1) Gauge Theory

We now introduce a gauge action to our notion of field theory. The Hopf-Frobenius formalism of QFTs encodes gauge theory through (H_I, F_I). We take the gauge group (or gauge Lie algebra) to be H_I (more precisely we take either the group algebra or universal enveloping algebra). Just like with (H_{ST}, F_{ST}), the particular representation we choose will correspond to different physical phenomena.

For $U(1)$ gauge symmetry, our internal symmetry space F_I is one-dimensional. Say that F_I is spanned by the vector θ. Then $U(1)$ acts on the fields as

$$f \otimes e_i \otimes \theta \mapsto f \otimes e_i \otimes e^{i\omega}\theta. \tag{5.104}$$

Using the isomorphism $F_f \otimes F_{ST} \otimes F_I \cong F_f \otimes F_{ST}$, we can think of the action of $U(1)$ as scalar multiplication on $F_f \otimes F_{ST}$ by a complex phase.

Physics notation: In physics notation, the dimensions of F_{ST} are normally encoded in a subscript or superscript, so we would write the action of $U(1)$ as

$$\phi_\mu \mapsto e^{i\omega}\phi_\mu.$$

In the case that ϕ is a scalar field, the index is suppressed, so $\phi \mapsto e^{i\omega}\phi$.

Connection: Our gauge connection will be a map

$$\nabla : F_I \to F_f \otimes F_{ST} \otimes F_I.$$

Applying our isomorphism $F_I \cong k$, where k is the base field, our connection can be thought of as a map

$$\nabla : k \to F_f \otimes F_{ST}. \tag{5.105}$$

This is the same data as an element of $F_f \otimes F_{ST}$, which we think of as a vector field. This element (or **4-vector field** in the case that F_{ST} is four-dimensional) is called the **electromagnetic four-potential**.

$\boxed{\text{Physics notation:}}$ The electromagnetic four-potential is normally denoted A_μ. In components, we have

$$\nabla\theta = \sum A_\mu(x) \otimes e_\mu \otimes \theta.$$

$\boxed{\text{Covariant derivative:}}$ The covariant derivative corresponding to A_μ will be

$$D_\nabla \phi = \sum_j \partial_j \phi \otimes e_j \otimes \theta + \sum_i \phi A_i(x) \otimes e_i \otimes \theta.$$

Using the isomorphism $F_I \cong k$, we can view the covariant derivative as

$$D_\nabla \phi = \sum_i \partial_i \phi \otimes e_i + \phi(x) A_i(x) \otimes e_i.$$

$\boxed{\text{Directional derivative:}}$

$$\langle v, D_\nabla \phi \rangle_2 = \sum_i v_i \partial_i \phi + \phi(x) v_i A_i(x).$$

It follows that

$$\nabla_i \phi = \partial_i \phi + \phi(x) A_i(x),$$

and so

$$\begin{aligned}
\nabla_a \nabla_b \phi &= \partial_a \partial_b \phi + \partial_a(\phi(x) A_b(x)) + (\partial_b \phi) A_a(x) + \phi(x) A_b(x) A_a(x) \\
&= \partial_a \partial_b \phi + (\partial_a \phi) A_b(x) + \phi \partial_a A_b + (\partial_b \phi) A_a(x) + \phi(x) A_b(x) A_a(x).
\end{aligned}$$

Curvature: It follows from the above expression for the directional derivative that

$$[\nabla_a, \nabla_b]\phi = (\partial_a A_b - \partial_b A_a)\phi. \tag{5.106}$$

The term $(\partial_a A_b - \partial_b A_a)$ is usually denoted F_{ab}, and called the **electromagnetic tensor**:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Physics notation: Physicists will denote the directional derivative as D_μ, where μ corresponds to our index over the dimensions of spacetime. A version of our directional derivative (which is also called the *covariant derivative* in physics) which may be more familiar to physicists is

$$D_\mu\phi = \partial_\mu\phi + \phi(x)A_\mu(x).$$

5.4.2 $SU(3)$ Gauge Theory

The Lie group $SU(3)$ is the gauge group for quantum chromodynamics. In this case, F_I is three-dimensional.

Connection: We have $\nabla : F_I \to F_f \otimes F_{ST} \otimes F_I$. The gauge fields have the form

$$\nabla\theta_i = \sum G_{ij}^k(x) \otimes e_j \otimes \theta_k. \tag{5.107}$$

Physics notation: The functions $G_{ij}^k(x)$ are called the **gluon field**, and physicists will often package together the G_{ij}^k and θ_k tensor factors, denoting the field by

$$\mathbf{G}_\mu$$

where μ runs over the four spacetime dimensions, and in each dimension we have an eight-dimensional vector field $G_\ell(x)$. The eight dimensions come from the two indices i, k taking values in $\{1, 2, 3\}$ (we lose one dimension because of the conditions making the action of $SU(3)$ into an irreducible representation).

$\boxed{\text{Covariant derivative:}}$

$$D_\nabla \varphi = \sum_{ij} \partial_j \varphi_i \otimes e_j \otimes \theta_i + \sum_{ijk} \varphi_i(x) G^k_{ij}(x) \otimes e_j \otimes \theta_k.$$

Or, expressed in a form more familiar to physicists, we have, in each dimension e_μ,

$$\begin{aligned} D_\mu \varphi &= \sum_i \partial_\mu \varphi_i \otimes \theta_i + \sum_{ik} \varphi_i(x) G^k_{i\mu}(x) \otimes \theta_k \\ &= \partial_\mu \varphi + G^k_\mu \cdot \varphi \end{aligned}$$

where, in the last line we are implicitly considering φ to be a three-dimensional vector.

5.5 Wightman Axioms

The Wightman axioms are a way of axiomatically defining a quantum field theory. The axioms are numerous and put very stringent requirements on what is considered a quantum field theory. The benefit is that from these axioms, one can prove that a Wightman QFT has physically desirable properties. For example it satisfies CPT symmetry (that is, there is a symmetry under inverting all three of (1) charge, (2) parity and (3) the flow of time, at once).

In this section we will present the Wightman axioms and then show how they can be reinterpreted as axioms on a system of three Hopf-Frobenius modules. Following Wightman and Streater's presentation in [35], we collect the axioms into several subcategories.

5.5.1 Relativistic Quantum Mechanics Axioms

These axioms assure that a Wightman QFT reproduces ordinary (relativistic) quantum mechanics.

1. $\boxed{\text{Hilbert Space:}}$ There is a separable Hilbert space X whose rays (elements of the corresponding projective space) are quantum states.

2. $\boxed{\text{Evolution:}}$ There is a continuous unitary representation of $SL(2,\mathbb{C})$ (which is the double cover of the restricted Lorentz group) on X, which we denote $U(L,a)$ (using the parametrization where each Lorentz transformation corresponds to a rotation L and a boost a).

3. $\boxed{\text{Spectral Condition:}}$ We can write $U(1,a) = \exp(iP)$ for some matrix P. Furthermore, the eigenvalues of P satisfy $p_0^2 - \sum_{i=1}^{3} p_i^2 > 0$ and $p_0 > 0$.

4. $\boxed{\text{Vacuum:}}$ There is a state of X, denoted $|0\rangle$, which is the unique state invariant under the action of $U(L,a)$. We call this the **vacuum state** of the theory.

5.5.2 Field Axioms

5. $\boxed{\text{Dense subset:}}$ There is a fixed dense subspace $D \subseteq X$ which contains the vacuum state, and is taken to itself by $U(L,a)$.

6. $\boxed{\text{Operator-valued Distributions:}}$ Components ϕ_i of the quantum field ϕ are operator-valued distributions on a Schwartz space:

$$\phi_i : \mathcal{S} \to B(D).$$

In physics we often work with unbounded operators. Given a Hermitian unbounded operator on D, it's possible for it to have multiple distinct extensions to Hermitian operators on the whole Hilbert space. So D has to be large enough to specify an extension uniquely.

5.5.3 Field Transformations

7. $\boxed{\text{Transformations:}}$ There is an action of $SL(2)$ on the fields, $S(L)$ such that

$$U(L,a)\phi(x)U(L,a)^{-1} = S(L)\phi(L^{-1}(x-a))$$

8. $\boxed{\text{Cyclic Vacuum:}}$ The image of polynomials in $\phi_i(f)$ on $|0\rangle$ is dense in X.

5.5.4 Microscopic Causality:

9. $\boxed{\text{Causality:}}$ If $f, g \in \mathcal{S}$ have spacelike supports (ie. $f(x)g(y) = 0$ for all x, y with $d(x,y) \geq 0$) then

$$[\phi_i(f), \phi_j(g)]_\pm(v) = 0 \text{ for all } v \in D,$$

where $[-,-]_\pm$ means take the commutator or anticommutator, depending on whether $\phi_i(f), \phi_j(g)$ are bosonic or fermionic operators, respectively.

5.5.5 Wightman Axioms for Hopf-Frobenius QFT

We now re-express some of the above axioms in terms of the structure of a Hopf-Frobenius QFT. In what follows we consider a HFQFT $(F_f, F_{ST}, F_I, H_f, H_{ST}, H_I)$.

Axioms:

1. Hilbert space, states, evolution: F_I is a separable Hilbert space and there is an action of H_{ST} on F_I which turns (H_{ST}, F_I) into a Hopf-Frobenius module.

2. Vacuum: There is an element $v_0 \in F_I$, which is the unique element preserved by H_{ST} and satisfies

$$\varepsilon(v_0 x) = \varepsilon(x)$$

for all $x \in F_I$.

3. Field Transformations: There is a Hopf-Frobenius module structure on $(H_{ST}, F_f \otimes F_I \otimes F_{ST})$.

Chapter 6

Operads and Properads

6.1 Introduction

Operads are a way of encoding algebraic structure by organizing operations like addition, Lie brackets, and so forth, into collections of *n-ary operations* $\mathcal{O}(n)$, which satisfy certain compatibility relations. Algebras over an operad are something like representations of a group: while a group is an abstract mathematical structure required to satisfy certain axioms, a representation of a group realizes the group explicitly as a subgroup of the general linear group on some vector space. Similarly, operads are abstract structures which can be realized explicitly in different categories.

It turns out that algebraic quantum field theories (AQFTs) can be expressed as algebras over a certain operad. We will show that AQFTs whose algebras of observables are Laplace Hopf algebras can be expressed as the algebras over a particular properad.

$\boxed{\text{Compatibility conditions:}}$ The compatibility conditions that operads have to satisfy (associativity, equivariance, and unitality) are fairly straightforward to guess, but are quite involved to write down explicitly. We refer the reader to [24, Chapter 5] or [41, Chapter 11] for the full description of the compatibility conditions of operads.

Definition 6.1.1: Non-Symmetric Operad in Sets

A **non-symmetric operad in Sets** is a collection of sets $\mathcal{O} = \{\mathcal{O}(n)\}_{n\geq 0}$ with a specified element $1 \in \mathcal{O}(1)$ and composition maps

$$\circ_{[n_1 \ldots n_k | n]} : \mathcal{O}(k) \times \mathcal{O}(n_1) \times \ldots \times \mathcal{O}(n_k) \to \mathcal{O}(n_1 + \ldots + n_k)$$

such that

- $f \circ (1, 1, \ldots, 1) = 1 \circ f = f$ for all $f \in \mathcal{O}(n)$.
- $f \circ (g_1 \circ (h_{1,1}, \ldots h_{1,k_1}), \ldots, g_n \circ (h_{n,1}, \ldots, h_{n,k_n})) = (f \circ (g_1, \ldots, g_n)) \circ (h_{1,1}, \ldots, h_{n,k_n})$.

Terminology: Elements of $\mathcal{O}(n)$ are called n-ary operations, and are interpreted as functions that have n inputs and one output.

Variations: Numerous flavours of operad have been introduced throughout the literature as variations on the above definition. To ease understanding, we will first consider the above notion of operad before considering any variations.

Definition 6.1.2: Morphism of Set Operads

Let $\mathcal{O}, \mathcal{P}$ be set operads. A **morphism of operads** $\phi : \mathcal{O} \to \mathcal{P}$ is a collection of set functions $\{\phi_n : \mathcal{O}(n) \to \mathcal{P}(n)\}$ such that

$$\phi(1_{\mathcal{O}}) = 1_{\mathcal{P}}$$
$$\phi(f \circ (g_1, \ldots, g_n)) = \phi(f) \circ (\phi(g_1), \ldots, \phi(g_n)).$$

$\boxed{\text{Intuition:}}$ We can think of these conditions as the operadic version of the conditions on k-algebra homomorphisms: the first condition is the preservation of the unit, the second is that composition commutes with the homomorphism.

Definition 6.1.3: Algebra over an Operad

Let $\mathcal{O}$ be a set operad. An **algebra over $\mathcal{O}$** is a set X and set maps

$$\theta_n : \mathcal{O}(n) \times X^n \to X \tag{6.108}$$

which have to satisfy associativity and unitality.

By picking out a particular set and collection of operations θ_n, the abstract operad $\mathcal{O}$ becomes realized as a specific algebraic structure.

Example 6.1.4: The Associative Operad in Set

The associative operad is the operad whose sets of operations are all singletons:

$$\mathcal{A}(n) = \{\mu_n\}, \qquad \text{for all } n \geq 0. \tag{6.109}$$

The main feature of the associative operad is that algebras over it will be the same thing as monoids. Let X be an algebra over the operad. We consider the operations on X:

Degree 0: μ_0 will become the unit element $\theta_0(\mu_0) : \{*\} \to X$.

Degree 1: μ_1 will become the identity map $\theta_1(\mu_1) : X \to X$.

Degree 2: μ_2 will become the multiplication $\theta_2(\mu_2) : X \times X \to X$.

Higher arity: The fact that we only have a single map in each degree will force certain compatibility conditions to hold. For example, there are two ways of composing maps in $\mathcal{A}$ in order to produce something with 3 inputs and one output: $\mu_2 \circ (\mu_2, 1)$ and $\mu_2 \circ (1, \mu_2)$. Since $\mathcal{A}(3)$ is a singleton, both compositions must be equal:

$$\mu_2 \circ (\mu_2, 1) = \mu_2 \circ (1, \mu_2). \tag{6.110}$$

This forces the multiplication on X to be associative. Similarly, the higher $\mathcal{A}(n)$ being singletons forces all possible bracketings of products in X to be equal. Similarly, one can show that the degree 0 term must behave as a unit. Thus algebras over the associative operad are precisely monoids.

Symmetric operad: We often want our algebraic structure to satisfy certain compatibility conditions which involve swapping terms around. For example commutativity says that $xy = yx$, or $xy = xy \cdot \sigma_2$, where σ_2 is the permutation which swaps x and y. Introducing an action of the symmetric group on n elements for each $\mathcal{O}(n)$ (and requiring certain equivariance conditions) produces the notion of a **symmetric operad**.

Enriched operad: We could also ask that our $\mathcal{O}(n)$ have some additional (or alternative) structure to being sets. For example, monoids in vector spaces are the same thing as k-algebras. So if we define a version of the associative operad in vector spaces, its algebras will be associative algebras. These more general operads are called **enriched operads**.

Coloured operad: Finally, we might want to allow our operad algebras to involve a choice of more than one object X. If we index our operad operations by a set C, we arrive at the notion of a **coloured operad**.

We allow all three of these extra features in the more general notion of operad which we now consider.

6.2 More General Operads

Definition 6.2.1: Coloured Symmetric Sequence

Let C be a set and $\mathcal{V}$ be a symmetric monoidal category. A **symmetric sequence in $\mathcal{V}$ with colours C** is a functor

$$\mathcal{O} : S[C] \times C \to \mathcal{V} \tag{6.111}$$

where $S[C]$ is the groupoid whose objects are finite sequences in C and morphisms are permutations.

Notation: Given a string $\alpha_1 \dots \alpha_n$ in $S[C]$, we will use the shorthand notation

$$\bar{\alpha} \equiv \alpha_1 \dots \alpha_n.$$

Interpretation: A symmetric sequence applied to an element

$$(\bar{\alpha}, \beta) \in S[C] \times C$$

corresponds to the object of operations from $\bar{\alpha}$ to β: $\mathcal{O}[\bar{\alpha} \mid \beta]$.

Definition 6.2.2: Coloured Operad

Let $\mathcal{V}$ be a symmetric monoidal category and C be a set of colours. A ***C*-coloured operad in $\boldsymbol{\mathcal{V}}$**, $\mathcal{O}$, is a coloured symmetric sequence $\mathcal{O} : S[C] \times C \to \mathcal{V}$ along with composition maps γ, and units 1_β

$$\begin{gathered} \gamma : \mathcal{O}[\alpha_1 \ldots \alpha_n \mid \beta] \otimes \bigotimes_{i=1}^{n} \mathcal{O}[\bar{\omega}_i \mid \alpha_i] \to \mathcal{O}[\bar{\omega} \mid \beta] \\ 1_\beta : I \to \mathcal{O}[\beta \mid \beta] \end{gathered} \tag{6.112}$$

for each choice of $\alpha_i, \beta, \bar{\omega}_i$, where I is the unit in $\mathcal{V}$. The composition maps and actions of S_n have to satisfy compatibility relations for associativity, unitality, and equivariance.

Definition 6.2.3: Algebra over an Operad

Let $\mathcal{V}$ be a symmetric monoidal category. An **algebra over a $\boldsymbol{\mathcal{V}}$-operad $\boldsymbol{\mathcal{O}}$** (also called an **$\mathcal{O}$-algebra**) is a set of objects in $\mathcal{V}$ indexed by the colours $\alpha \in C$: $\{X_\alpha \mid X \in \mathrm{Obj}(\mathcal{V}), \alpha \in C\}$ and composition maps

$$\theta_{\alpha_1,\ldots,\alpha_n|\beta} : \mathcal{O}[\alpha_1, \ldots, \alpha_n \mid \beta] \otimes X_{\alpha_1} \ldots \otimes X_{\alpha_n} \to X_\beta \tag{6.113}$$

which have to satisfy associativity, unitality, and equivariance conditions.

$\boxed{\text{Notation:}}$ Given a selection of objects X_{α_i} or morphisms f_{α_i}, we use the shorthand

$$X_{\bar{\alpha}} \equiv X_{\alpha_1} \otimes \ldots \otimes X_{\alpha_n}$$
$$f_{\bar{\alpha}} \equiv f_{\alpha_1} \otimes \ldots \otimes f_{\alpha_n}.$$

Definition 6.2.4: Morphisms of Operad Algebras

Let $\mathcal{O}$ be an operad in the symmetric monoidal category $\mathcal{V}$. A **morphism of $\mathcal{O}$-algebras**, $f : (\theta, \{X_\alpha\}) \to (\phi, \{Y_\alpha\})$ is a collection of maps $f_\alpha : X_\alpha \to Y_\alpha$ such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{O}[\bar{\alpha} \mid \beta] \otimes X_{\bar{\alpha}} \arrow[r, "\theta"] \arrow[d, "1\otimes f_{\bar{\alpha}}"'] & X_\beta \arrow[d, "f_\beta"] \\
\mathcal{O}[\bar{\alpha} \mid \beta] \otimes Y_{\bar{\alpha}} \arrow[r, "\phi"'] & Y_\beta.
\end{tikzcd}$$

$\boxed{\text{Generators / Relations:}}$ Like many algebraic structures, operads can be expressed in terms of presentations of generators and relations. We refer the reader to [24, Section 5.5] or [41, Part 4] for more details. The main construction in the next section we will be produced in terms of generators and relations.

6.3 Operad of Operad Algebras

Even though we would like to consider operads in more general categories than SET, it is useful to be able to work element-wise when the opportunity is available. We first demonstrate how we can often perform operadic constructions in a SET-like way, by utilizing coproducts of the monoidal unit, before applying this idea to construct an operad whose algebras are functors

$$F : \mathcal{C} \to \mathcal{O}\text{-Alg}$$

between a given small category $\mathcal{C}$ and the category of algebras over a given operad $\mathcal{O}$. For our warm-up example, we will consider the case where $\mathcal{O}$ is the trivial operad.

Distributive monoidal: In what follows we will need to assume that in our monoidal category $\mathcal{V}$, the tensor product distributes over finite coproducts. This is not a substantial limitation. Categories which satisfy this property include all distributive categories (viewed as cartesian monoidal categories), abelian groups, R-modules for a commutative ring R, and pointed topological spaces with monoidal product the smash product and coproduct the wedge sum.

Example 6.3.1: Operad Construction: Coproducts of Units

Let $\mathcal{V}$ be a monoidal category with all small coproducts, whose tensor product distributes over finite coproducts, and let $\mathcal{D}$ be a small category. There is an operad $\mathcal{O}$ whose algebras are precisely the functors

$$F : \mathcal{D} \to \mathcal{V}.$$

Note: Let $\mathcal{I}$ be the trivial operad: it has a single colour, and $\mathcal{I}[1] = I$ and $\mathcal{I}[n] = \varnothing$ for all $n \neq 1$, where I is the tensor unit in $\mathcal{V}$. Algebras over this operad are just choices of objects in $\mathcal{V}$, and morphisms of $\mathcal{I}$-algebras are morphisms in $\mathcal{V}$ between the chosen objects. Thus

$$\mathcal{I}\text{-Alg} \cong \mathcal{V}.$$

So this example is of the desired form.

$\boxed{\text{Colours:}}$ To construct our desired operad, we take our colours C to be the objects of $\mathcal{D}$.

$\boxed{\text{Operations:}}$ For each pair of colours $[A \mid B]$, where A, B are objects in $\mathcal{D}$, we define the objects of our operad to be

$$\mathcal{O}[A \mid B] \equiv \bigsqcup_{f:A\to B} I.$$

For the sake of readability, we will index the units that appear in the coproduct by the morphisms that the coproduct is over: I_f, for

$$f : A \to B.$$

$\boxed{\text{Units:}}$ For each colour A, our operad unit is defined to be

$$1_A : I_{id_A} \to \mathcal{O}[A \mid A],$$

the map into the coproduct corresponding to the identity morphism $id_A : A \to A$ in $\mathcal{D}$.

$\boxed{\mathcal{O}\text{-algebras:}}$ Algebras over the operad $\mathcal{O}$ will pick out objects F_A for each colour A, and maps

$$\theta : \mathcal{O}[A \mid B] \otimes F_A \to F_B.$$

Since $\mathcal{O}[A \mid B]$ is a coproduct and the tensor product distributes over the coproduct by assumption, the map θ is the same data as maps

$$(I_f \otimes F_A) \cong F_A \to F_B$$

for all $f : A \to B$. We will denote these maps by

$$F_f : F_A \to F_B.$$

Then the following diagram commutes for all $f : A \to B$

$$\begin{tikzcd}
I_f \otimes F_A \arrow[r, "f\otimes 1"] \arrow[d, "\lambda"'] & \mathcal{O}[A \mid B] \otimes A \arrow[d, "\theta"] \\
F_A \arrow[r, "F_f"'] & F_B.
\end{tikzcd}$$

This is almost enough to reproduce all functors $\mathcal{D} \to \mathcal{V}$, but we need to make sure that F preserves composites and identities. To do so, we impose certain relations.

$\boxed{\text{Relations:}}$ If we wish to impose relations on our maps $F_A \to F_B$, it's enough to impose them in our operad $\mathcal{O}$. Consider the associativity condition on operad algebras:

$$\begin{tikzcd}
\mathcal{O}[A \mid B] \otimes \mathcal{O}[X \mid A] \otimes F_X \arrow[r, "\gamma\otimes 1"] \arrow[d, "1\otimes\theta"'] & \mathcal{O}[X \mid B] \otimes F_X \arrow[d, "\theta"] \\
\mathcal{O}[A \mid B] \otimes F_A \arrow[r, "\theta"'] & F_B.
\end{tikzcd}$$

If we add a tensor product of monoidal units to our diagram, we can draw

$$\begin{tikzcd}
F_X \arrow[r] & I_f \otimes I_g \otimes F_X \arrow[d, "f \otimes g \otimes 1"'] & \\
& \mathcal{O}[A \mid B] \otimes \mathcal{O}[X \mid A] \otimes F_X \arrow[r, "\gamma \otimes 1"] \arrow[d, "1 \otimes \theta"'] & \mathcal{O}[X \mid B] \otimes F_A \arrow[d, "\theta"] \\
& \mathcal{O}[A \mid B] \otimes F_A \arrow[r, "\theta"'] & F_B.
\end{tikzcd}$$

The path that travels down the left side of the square is

$$\begin{aligned}
\theta(1 \otimes \theta)(f \otimes g \otimes 1)\lambda^{-1} &= \theta(f \otimes F_g) \\
&= \theta(f \otimes 1)(1 \otimes F_g) \\
&= F_f \circ F_g
\end{aligned}$$

where λ is the left unitor. The other path is

$$\begin{aligned}
\theta(\gamma \otimes 1)(f \otimes g \otimes 1)\lambda^{-1} &= \theta((f \circ g) \otimes 1) \\
&= F_{f \circ g}.
\end{aligned}$$

Thus every operad algebra will satisfy

$$F_f \circ F_g = F_{f \circ g}.$$

To check that F preserves identities, we use the unitality condition on $\mathcal{O}$-algebras:

$$\begin{tikzcd}
I \otimes F_A \arrow[d, "1_A \otimes 1"'] \arrow[dr, "\lambda"] & \\
\mathcal{O}[A \mid A] \otimes F_A \arrow[r, "\theta"'] & F_A.
\end{tikzcd}$$

Thus $id_{F_A} = \theta \circ (1_A \otimes 1) \circ \lambda^{-1} = F_{id_A}$, and so F preserves identities. Thus our $\mathcal{O}$-algebras are precisely the functors

$$F : \mathcal{D} \to \mathcal{V}.$$

Furthermore, it follows that if $f \circ g = h \circ k$, then their corresponding maps F will also match:

$$F_f \circ F_g = F_h \circ F_k.$$

Thus relations on the morphisms f in the category $\mathcal{V}$ will produce the same relations on the maps F_f induced by the operad algebras. We will use this fact in the following proposition.

We will use the above type of construction in the following proof.

Proposition 6.3.2: The Operad of Operad Algebras

Let $\mathcal{V}$ be a symmetric monoidal category with all small coproducts, whose tensor product distributes over finite coproducts, and let $\mathcal{O}$ be a coloured operad in $\mathcal{V}$. For any small category, $\mathcal{C}$, there exists an operad $\mathcal{O}^{\mathcal{C}}$ in $\mathcal{V}$ whose algebras are the functors

$$F : \mathcal{C} \to \mathcal{O}\text{-Alg}.$$

Proof. $\boxed{\text{Colours:}}$ Let $R_{\mathcal{O}}$ be the set of colours of the operad $\mathcal{O}$. The colours of $\mathcal{O}^{\mathcal{C}}$ will be the set $\mathrm{Obj}(\mathcal{C}) \times R_{\mathcal{O}}$, which we think of as a copy of the colours of $\mathcal{O}$ at each object of $\mathcal{C}$. For the sake of space, we will denote the colours of $\mathcal{O}^{\mathcal{C}}$ as A_α, where $A \in \mathrm{Obj}(C)$ and $\alpha \in R_{\mathcal{O}}$. Thus the coloured symmetric sequence $\mathcal{O}^{\mathcal{C}}$ will act on $[A_{1,\alpha_1} \dots A_{n,\alpha_n} \mid B_\beta]$.

$\boxed{\mathcal{O}^{\mathcal{C}} \text{ algebras:}}$ Once we've constructed our operad, an algebra over it will assign to each A_α an object $G(A_\alpha)$ in $\mathcal{V}$. We'll also have operation maps

$$\theta : \mathcal{O}^{\mathcal{C}}[A_{1,\alpha_1} \dots A_{n,\alpha_n} \mid B_\beta] \otimes \bigotimes_i G(A_{\alpha_i}) \to G(B_\beta). \tag{6.114}$$

$\boxed{\text{Inter-object Operations:}}$ Given two distinct objects $A, B \in \mathrm{Obj}(\mathcal{C})$ and a colour $\alpha \in R_{\mathcal{O}}$, we define

$$\mathcal{O}^{\mathcal{C}}_{\mathrm{gen}}[A_\alpha \mid B_\alpha] \equiv \bigsqcup_{f \in \mathrm{Mor}_{\mathcal{C}}(A,B)} I_f \tag{6.115}$$

where I_f are all the monoidal unit in $\mathcal{V}$, but we've indexed them by morphisms in $\mathcal{C}$ as in Example 6.3.1, to keep track of what they each are meant to encode. Given an algebra over our operad, the operation maps

$$\theta : \mathcal{O}^{\mathcal{C}}_{\mathrm{gen}}[A_\alpha \mid B_\alpha] \otimes G(A_\alpha) \to G(B_\alpha)$$

will correspond to maps

$$\mu^\alpha_f : G(A_\alpha) \to G(B_\alpha),$$

as in Example 6.3.1 (in which we denoted such maps F_f).

$\boxed{\text{Copies of } \mathcal{O}\text{:}}$ At each object of $\mathcal{C}$ we need to both reconstruct the operad $\mathcal{O}$ and obtain the endomorphisms in $\mathcal{C}$. We take

$$\mathcal{O}^{\mathcal{C}}_{\mathrm{gen}}[A_{\alpha_1} \dots A_{\alpha_n} \mid A_\beta] \equiv \mathcal{O}[\alpha_1 \dots \alpha_n \mid \beta] \tag{6.116}$$

when $n \geq 2$ or $n = 0$, and

$$\mathcal{O}^{\mathcal{C}}_{\text{gen}}[A_\alpha \mid A_\alpha] \equiv \mathcal{O}[\alpha \mid \alpha] \sqcup \bigsqcup_{f \in \mathrm{Mor}_{\mathcal{C}}(A,A)} I_f. \tag{6.117}$$

$\boxed{\text{Compositions:}}$

1. For compositions involving only copies of $\mathcal{O}^{\mathcal{C}}_{\text{gen}}[A_{\bar{\alpha}} \mid A_\beta]$ we copy the composition maps from $\mathcal{O}$.

2. For operations coming from the morphisms in $\mathcal{C}$, following Example 6.3.1, it's enough to impose that

$$\begin{aligned} \mu^\alpha_f \circ \mu^\alpha_g &= \mu^\alpha_{f \circ g} \\ \mu^\alpha_{id_A} &= 1^\alpha, \end{aligned}$$

   where 1^α is the identity operation associated to the colour α.

3. Finally, we need ensure that all of the maps μ^α_f coming from morphisms in $\mathcal{C}$ behave like morphisms of $\mathcal{O}$-algebras. When we take an algebra over $\mathcal{O}^{\mathcal{C}}$, each A_α will be associated to an object in $\mathcal{V}$, say $G(A_\alpha)$. Now, the diagram we need to commute, for any $f : A \to B$ in $\mathcal{C}$, is

$$\begin{tikzcd} \mathcal{O}^{\mathcal{C}}_{\text{gen}}[A_{\bar{\alpha}} \mid A_\beta] \otimes G(A_{\bar{\alpha}}) \arrow[r, "\theta"] \arrow[d, "\nu^{\bar{\alpha}} \otimes \mu^{\bar{\alpha}}_f"'] & G(A_\beta) \arrow[d, "\mu^\beta_f"] \\ \mathcal{O}^{\mathcal{C}}_{\text{gen}}[B_{\bar{\alpha}} \mid B_\beta] \otimes G(B_{\bar{\alpha}}) \arrow[r, "\theta"'] & G(B_\beta), \end{tikzcd}$$

where we've identified $\mathcal{O}^{\mathcal{C}}_{\text{gen}}[A_{\bar{\alpha}} \mid A_\beta]$ and $\mathcal{O}^{\mathcal{C}}_{\text{gen}}[B_{\bar{\alpha}} \mid B_\beta]$ with $\nu^{\bar{\alpha}}$, since they are both equal to $\mathcal{O}[\bar{\alpha} \mid \beta]$ by definition. If we expand out the above diagram in terms of f^{α_i}, we have

$$\begin{tikzcd}
\mathcal{O}^{\mathcal{C}}_{\text{gen}}[A_{\bar{\alpha}} \mid A_\beta] \otimes G(A_{\bar{\alpha}}) \arrow[r] \arrow[d] & I \otimes \mathcal{O}^{\mathcal{C}}_{\text{gen}}[A_{\bar{\alpha}} \mid A_\beta] \otimes G(A_{\bar{\alpha}}) \arrow[d, "f\otimes 1"] \\
\mathcal{O}^{\mathcal{C}}_{\text{gen}}[A_{\bar{\alpha}} \mid A_\beta] \bigotimes_i I_{f^{\alpha_i}} \otimes G(A_{\bar{\alpha}}) \arrow[d, "\nu^{\bar{\alpha}} \bigotimes_i f^{\alpha_i} \otimes 1"'] & \mathcal{O}^{\mathcal{C}}_{\text{gen}}[A_\beta \mid B_\beta] \otimes \mathcal{O}^{\mathcal{C}}_{\text{gen}}[A_{\bar{\alpha}} \mid A_\beta] \otimes G(A_{\bar{\alpha}}) \arrow[d, "1\otimes\theta"] \\
\mathcal{O}^{\mathcal{C}}_{\text{gen}}[B_{\bar{\alpha}} \mid B_\beta] \bigotimes_i \mathcal{O}^{\mathcal{C}}_{\text{gen}}[A_{\alpha_i} \mid B_{\alpha_i}] \otimes G(A_{\bar{\alpha}}) \arrow[d, "1\otimes\theta"'] & \mathcal{O}^{\mathcal{C}}_{\text{gen}}[A_\beta \mid B_\beta] \otimes G(A_\beta) \arrow[d, "\theta"] \\
\mathcal{O}^{\mathcal{C}}_{\text{gen}}[B_{\bar{\alpha}} \mid B_\beta] \otimes G(B_{\bar{\alpha}}) \arrow[r, "\theta"'] & G(B_\beta)
\end{tikzcd}$$

Now we apply the associativity condition of operad algebras. We have two compositions of the form $\theta(1 \otimes \theta)$, which we can re-express as $\theta(\gamma \otimes 1)$. This lets us re-write the above diagram as

$$\begin{tikzcd}
\mathcal{O}^{\mathcal{C}}_{\text{gen}}[A_{\bar{\alpha}} \mid A_\beta] \otimes G(A_{\bar{\alpha}}) \arrow[r] \arrow[d] & I \otimes \mathcal{O}^{\mathcal{C}}_{\text{gen}}[A_{\bar{\alpha}} \mid A_\beta] \otimes G(A_{\bar{\alpha}}) \arrow[d, "f\otimes 1"] \\
\mathcal{O}^{\mathcal{C}}_{\text{gen}}[A_{\bar{\alpha}} \mid A_\beta] \bigotimes_i I_{f^{\alpha_i}} \otimes G(A_{\bar{\alpha}}) \arrow[d, "\nu^{\bar{\alpha}} \bigotimes_i f^{\alpha_i} \otimes 1"'] & \mathcal{O}^{\mathcal{C}}_{\text{gen}}[A_\beta \mid B_\beta] \otimes \mathcal{O}^{\mathcal{C}}_{\text{gen}}[A_{\bar{\alpha}} \mid A_\beta] \otimes G(A_{\bar{\alpha}}) \arrow[d, "\gamma\otimes 1"] \\
\mathcal{O}^{\mathcal{C}}_{\text{gen}}[B_{\bar{\alpha}} \mid B_\beta] \bigotimes_i \mathcal{O}^{\mathcal{C}}_{\text{gen}}[A_{\alpha_i} \mid B_{\alpha_i}] \otimes G(A_{\bar{\alpha}}) \arrow[r, "\gamma\otimes 1"] & \mathcal{O}^{\mathcal{C}}_{\text{gen}}[A_{\bar{\alpha}} \mid B_\beta] \otimes G(A_{\bar{\alpha}}) \arrow[d, "\theta"] \\
 & G(B_\beta)
\end{tikzcd}$$

But now the commutativity condition no longer depends on θ, so we can impose it inside of the operad $\mathcal{O}^{\mathcal{C}}_{\text{gen}}$ itself. Forcing the upper square of this last diagram to commute will thus assure that all of the $f : A \to B$ become morphisms of $\mathcal{O}$-algebras. Imposing the above requirements produces our desired operad $\mathcal{O}^{\mathcal{C}}$.

It follows that algebras over the operad $\mathcal{O}^{\mathcal{C}}$ will be all of the functors from $\mathcal{C}$ into $\mathcal{O}$-algebras. $\square$

6.4 Properads

The coalgebraic structure of Hopf algebras and Hopf-Frobenius modules and the way that it interacts with the algebraic structure have played a crucial role in our formulation of quantum field theory. However operads on their own do not have the capacity to encode co-operations. Similarly, co-operads can encode coalgebraic structure, but not algebraic. Properads are mathematical objects which can encode both algebraic and coalgebraic operations and compositions between them.

Updating our notions from the operadic to properadic set-up is straightforward, and essentially consists of replacing the operation sets $\mathcal{O}[\bar{\alpha} \mid \beta]$ with

$$\mathcal{P}[\bar{\alpha} \mid \bar{\beta}],$$

where now both $\bar{\alpha}$ and $\bar{\beta}$ can be strings of length longer than one.

For more information on properads, see [18, Chapter 3], [17, Section 3], and [40, Section 2].

Definition 6.4.1: Coloured Properadic Symmetric Sequence

Let C be a set and $\mathcal{V}$ be a symmetric monoidal category. Then a **pr-operadic symmetric sequence in $\mathcal{V}$ with colours C** is a functor

$$\mathcal{P} : S[C] \times S[C] \to \mathcal{V} \tag{6.118}$$

where $S[C]$ is the groupoid whose objects are finite sequences in C and morphisms are permutations.

> We interpret $\mathcal{P}[\bar{\alpha} \mid \bar{\beta}]$ as an object of operations with inputs $\alpha_1, \ldots, \alpha_m$ and outputs $\beta_1, \ldots, \beta_n$.

Definition 6.4.2: Coloured Properad

Let $\mathcal{V}$ be a symmetric monoidal category and C be a set of colours. A **C-coloured properad in $\mathcal{V}$**, $\mathcal{P}$, is a coloured properadic symmetric sequence $\mathcal{P} : S[C] \times S[C] \to \mathcal{V}$ along with composition maps γ and units 1_β

$$\begin{gathered}\gamma_{[\bar{\alpha}]_1 \ldots [\bar{\alpha}]_n} : \mathcal{P}[(\bar{\alpha})_1 \ldots (\bar{\alpha})_n \mid \bar{\beta}] \otimes \bigotimes_{i=1}^{n} \mathcal{P}[\bar{\omega}_i \mid (\bar{\alpha})_n] \to \mathcal{P}[\bar{\omega} \mid \bar{\beta}] \\ 1_\beta : I \to \mathcal{P}[\beta \mid \beta],\end{gathered} \tag{6.119}$$

which have to satisfy compatibility relations for associativity, unitality, and equivariance.

Definition 6.4.3: Algebra over a Properad

Let $\mathcal{V}$ be a symmetric monoidal category. An **algebra in the category $\mathcal{V}$ over a properad $\mathcal{P}$** is a set of objects in $\mathcal{D}$ indexed by the colours $\alpha \in C$

$$\{X_\alpha \mid X \in \mathrm{Obj}(\mathcal{D}), \alpha \in C\},$$

and composition maps

$$\theta_{\bar{\alpha}|\bar{\beta}} : \mathcal{P}[\bar{\alpha} \mid \bar{\beta}] \otimes X_{\alpha_1} \otimes \ldots \otimes X_{\alpha_m} \to X_{\beta_1} \otimes \ldots \otimes X_{\beta_n} \tag{6.120}$$

which have to satisfy certain associativity, unitality, and equivariance conditions.

Definition 6.4.4: Morphisms of Properad Algebras

Let $X = (\theta, \{X_\alpha\})$, $Y = (\phi, \{Y_\alpha\})$ be algebras over the properad $\mathcal{P}$. Then a morphism $f : X \to Y$ is a collection of maps $f_\alpha : X_\alpha \to Y_\alpha$ such that the diagram

$$\begin{tikzcd}
\mathcal{P}[\bar{\alpha} \mid \bar{\beta}] \otimes X_{\bar{\alpha}} \arrow[r, "\theta"] \arrow[d, "1\otimes f_{\bar{\alpha}}"'] & X_{\bar{\beta}} \arrow[d, "f_{\bar{\beta}}"] \\
\mathcal{P}[\bar{\alpha} \mid \bar{\beta}] \otimes Y_{\bar{\alpha}} \arrow[r, "\phi"'] & Y_{\bar{\beta}}
\end{tikzcd}$$

commutes.

Example 6.4.5: Properad: Bialgebras

A properad encoding bialgebras can be constructed in a straightforward manner using generators and relations in the category of vector spaces.

For more information see [26, Section 5.4] and [27, Section 3.3].

We simply ask for generators $\{\mu, \Delta, \eta, \varepsilon)$ and impose associativity, unitality, coassociativity, counitality, and the bialgebra compatibility condition. This compatibility condition is the only relation that distinguishes this construction from those of the (co)associative operads, and it requires the use of the symmetric group S_2 action σ:

$$\Delta\mu \sim (\mu \otimes \mu)(1 \otimes \sigma \otimes 1)(\Delta \otimes \Delta).$$

Example 6.4.6: Properad: Laplace Bialgebras

We can modify the above construction to create an operad whose algebras are bialgebras with Laplace pairings. To do so, we add an extra generator, $L \in \mathcal{P}(2, 0)$ and impose the relations from Equation (3.51):

$$\begin{aligned} L \circ 1 \otimes \mu &\sim L \otimes L(1 \otimes \sigma \otimes 1)(\Delta \otimes 1 \otimes 1) \\ L \circ (\mu \otimes 1) &\sim (L \otimes L)(1 \otimes \sigma \otimes 1)(1 \otimes 1 \otimes \Delta) \\ L \circ (\eta \otimes 1) &\sim L \circ (1 \otimes \eta) \sim \varepsilon. \end{aligned}$$

We now sketch the proof of the analogous theorem to Proposition 6.3.2 in the case of properads.

Properad presentations In order to follow Proposition 6.3.2, we need to first show that properads have presentations given by generators and relations. We sketch the following argument graciously provided by Donald Yau. From [43, Corollary 11.28], for any set of colours C, the category of C-coloured properads is equivalent to the category of G-PROPs for a particular pasting scheme $G^{\uparrow}_{C}$. From [43, Theorem 14.1], every pasting scheme G has a corresponding operad U_G such that the algebras over U_G are precisely the G-PROPs. It follows that there is an operad $U_{G^{\uparrow}_{C}}$ whose algebras are C-coloured properads. Now the category of O-algebras has free objects, for any coloured operad O (see [39, Example 4.1.10 (1)]). Relations can be built from coequalizers. It follows that every properad has a presentation in terms of generators and relations.

Proposition 6.4.7: Properad of Properad Algebras

Let $\mathcal{P}$ be a coloured properad in a symmetric monoidal category $\mathcal{V}$ with all small coproducts, whose tensor product distributes over finite coproducts. For any small category $\mathcal{C}$, there exists a properad $\mathcal{P}^{\mathcal{C}}$ whose algebras are the functors

$$F : \mathcal{C} \to \mathcal{P}\text{-Alg}. \tag{6.121}$$

Proof. We sketch the proof below — it essentially follows the structure of Proposition 6.3.2.

$\boxed{\text{Colours:}}$ Let $C_{\mathcal{P}}$ be the set of colours of the properad $\mathcal{P}$. The colours of $\mathcal{P}^{\mathcal{C}}$ will be the set $\mathrm{Obj}(\mathcal{C}) \times C_{\mathcal{P}}$. As in Proposition 6.3.2, we denote by A_α the colour $(A, \alpha) \in \mathrm{Obj}(\mathcal{C}) \times C_{\mathcal{P}}$.

$\boxed{\text{Inter-object Operations:}}$ Given two distinct objects $A, B \in \mathrm{Obj}(\mathcal{C})$

and a colour $\alpha \in C_{\mathcal{P}}$, we choose

$$\mathcal{P}^{\mathcal{C}}_{\text{gen}}[A_\alpha \mid B_\alpha] \equiv \bigsqcup_{f:A\to B} I_f. \tag{6.122}$$

$\boxed{\text{Copies of } \mathcal{P}\text{:}}$ At each object of $\mathcal{C}$ we need to both reconstruct the properad $\mathcal{P}$ and obtain the endomorphisms in $\mathcal{C}$. We let $A_{\bar{\alpha}}$ denote the string of colours $A_{\alpha_1} \dots A_{\alpha_n}$. Now take

$$\mathcal{P}^{\mathcal{C}}_{\text{gen}}[A_{\bar{\alpha}} \mid A_{\bar{\beta}}] \equiv \mathcal{P}[\bar{\alpha} \mid \bar{\beta}] \tag{6.123}$$

when $m \neq 1$ or $n \neq 1$, and

$$\mathcal{P}^{\mathcal{C}}_{\text{gen}}[A_\alpha \mid A_\alpha] \equiv \mathcal{P}[\alpha \mid \alpha] \sqcup \bigsqcup_{f:A\to A} I_f. \tag{6.124}$$

$\boxed{\text{Compositions:}}$

1. For generators of the form $\mathcal{P}^{\mathcal{C}}_{\text{gen}}[A_\alpha \mid B_\alpha] \equiv \bigsqcup_{f:A\to B} I_f$, we use the composition from $\mathcal{C}$.

2. For compositions involving only copies of $\mathcal{P}^{\mathcal{C}}_{\text{gen}}[A_{\bar{\alpha}} \mid A_{\bar{\beta}}]$ we use the corresponding composition maps in $\mathcal{P}$.

3. To reproduce $\mathcal{P}$-algebra morphisms, we enforce

$$
\begin{tikzcd}
\mathcal{P}^{\mathcal{C}}_{\text{gen}}[A_{\bar{\alpha}} \mid A_{\bar{\beta}}] \otimes G(A_{\bar{\alpha}}) \arrow[r] \arrow[d] & I \otimes \mathcal{P}^{\mathcal{C}}_{\text{gen}}[A_{\bar{\alpha}} \mid A_{\bar{\beta}}] \otimes G(A_{\bar{\alpha}}) \arrow[d, "f\otimes 1"] \\
\mathcal{P}^{\mathcal{C}}_{\text{gen}}[A_{\bar{\alpha}} \mid A_{\bar{\beta}}] \bigotimes_i I_{f^{\alpha_i}} \otimes G(A_{\bar{\alpha}}) \arrow[d, "\nu^{\bar{\alpha}} \bigotimes_i f^{\alpha_i} \otimes 1"'] & \mathcal{O}^{\mathcal{C}}_{\text{gen}}[A_{\bar{\beta}} \mid B_{\bar{\beta}}] \otimes \mathcal{P}^{\mathcal{C}}_{\text{gen}}[A_{\bar{\alpha}} \mid A_{\bar{\beta}}] \otimes G(A_{\bar{\alpha}}) \arrow[d, "\gamma\otimes 1"] \\
\mathcal{P}^{\mathcal{C}}_{\text{gen}}[B_{\bar{\alpha}} \mid B_{\bar{\beta}}] \bigotimes_i \mathcal{P}^{\mathcal{C}}_{\text{gen}}[A_{\alpha_i} \mid B_{\alpha_i}] \otimes G(A_{\bar{\alpha}}) \arrow[r, "\gamma\otimes 1"] & \mathcal{P}^{\mathcal{C}}_{\text{gen}}[A_{\bar{\alpha}} \mid B_{\bar{\beta}}] \otimes G(A_{\bar{\alpha}}) \arrow[d, "\theta"] \\
 & G(B_{\bar{\beta}})
\end{tikzcd}
$$

□

6.5 Properad of Laplace Hopf Quantum Field Theories

$\boxed{\text{Big picture:}}$ AQFTs are functors $F : \mathcal{X} \to \mathcal{A}$, from a category encoding spacetime regions to some chosen algebraic category. The functor, or AQFT, is meant to encode the algebra of observables at each spacetime region.

In the literature, different algebraic categories $\mathcal{A}$ have been studied. Traditional choices for $\mathcal{A}$, like categories of C^* algebras or various monoids, don't come equipped with any notion of correlation function. Below we first review the standard notion of AQFT before proposing our version of an AQFT with correlation functions. We follow the description of AQFTs in [42].

Definition 6.5.1: Orthogonality Relation

Let $\mathcal{C}$ be a small category. An **orthogonality relation** $\perp$ on $\mathcal{C}$ is a relation on pairs of morphisms f, g in $\mathcal{C}$ with the same codomain, such that the relation satisfies the following:

1. If $f \perp g$ then $g \perp f$.
2. If $g \perp h$ then $fg \perp fh$ for all composable f.
3. If $g_1 \perp g_2$ then $g_1 h \perp g_2 k$ for all composable h, k.

Definition 6.5.2: Orthogonal Category

An **orthogonal category** is a small category $\mathcal{C}$ with an orthogonality relation $\perp$.

Definition 6.5.3: Algebraic Quantum Field Theory

Let $(\mathcal{C}, \perp)$ be an orthogonal category. An **algebraic quantum field theory** is a functor

$$F : \mathcal{C} \to \mathrm{Mon}(\mathcal{M})$$

to the category of monoids of some monoidal category $\mathcal{M}$, such that

$$\begin{tikzcd}
F(A) \otimes F(B) \arrow[r, "F(f)\otimes F(g)"] \arrow[d, "F(f)\otimes F(g)"'] & F(C) \otimes F(C) \arrow[r, "\tau"] & F(C) \otimes F(C) \arrow[d, "\mu"] \\
F(C) \otimes F(C) \arrow[rr, "\mu"'] & & F(C)
\end{tikzcd}$$

commutes whenever $f : A \to C$ and $g : B \to C$ satisfy $f \perp g$.

$\boxed{\text{Causality:}}$ The commutative diagram condition is called the **causality condition** and corresponds to the Wightman axiom that space-like separated quantum fields should commute (or anticommute for fermions).

In [6] an operad is constructed whose algebras are AQFTs. We can essentially reproduce their construction using Proposition 6.3.2: first note that monoids in a monoidal category are algebras over the associative operad. Proposition 6.3.2 then produces an operad whose algebras are functors of the correct shape,

$$F : \mathcal{C} \to \mathrm{Mon}(\mathcal{M}),$$

but may not satisfy causality. This auxillary operad $\mathcal{A}^{\mathcal{C}}$ was denoted $\mathcal{O}_C$ in [6, Definition 3.6]. In order to rectify this, the authors of [6] modify the construction of the operad $\mathcal{A}^{\mathcal{C}}$, by imposing additional relations that force the causality diagram to commute. This construction essentially relies on the fact that the category (and orthogonality relation) are fixed, and everything in the causality condition can be expressed in terms of the elements of the operad $\mathcal{A}^{\mathcal{C}}$: μ are operations in $\mathcal{A}$, f, g are morphisms in $\mathcal{C}$, and τ is the swap map, which corresponds to the action of S_2 on the operad.

$\boxed{\text{Properads:}}$ Since we know that Laplace bialgebras can be encoded as the algebras over a properad (Example 6.4.6), Proposition 6.4.7 gives us a properad whose algebras are functors

$$F : \mathcal{C} \to \mathcal{LB}(\mathcal{V})$$

where $\mathcal{LB}(\mathcal{V})$ is the category of ordinary bialgebras with Laplace pairings in the symmetric monoidal category $\mathcal{V}$.

Definition 6.5.4: Cluster Decomposition Condition

Let $(\mathcal{C}, \perp)$ be an orthogonal category and $\mathcal{V}$ be a symmetric monoidal category. Consider the category $\mathcal{LB}_{cc}$ of *cocommutative* Laplace bialgebras. A functor $F : \mathcal{C} \to \mathcal{LB}_{cc}(\mathcal{V})$ satisfies the **cluster decomposition condition** if

$$\begin{tikzcd}
F(A)^{\otimes m} \otimes F(B)^{\otimes n} \arrow[r, "F(f)^{\otimes m} \otimes F(g)^{\otimes n}"] \arrow[d, "\langle -,- \rangle_m \otimes \langle -,- \rangle_n"'] & F(C)^{\otimes m} \otimes F(C)^{\otimes n} \arrow[d, "\langle -,- \rangle_{m+n}"] \\
k \otimes k \arrow[r, "\mu_k"'] & k
\end{tikzcd}$$

commutes whenever $f : A \to C$ and $g : B \to C$ satisfy $f \perp g$, where $\langle -, - \rangle_{m+n}$ is the $(m+n)$-arity correlation function on $F(C)$, as described underneath Corollary 3.4.17.

Definition 6.5.5: Laplace Bialgebraic AQFT

Let $(\mathcal{C}, \perp)$ be an orthogonal category and $\mathcal{V}$ be a symmetric monoidal category. A **Laplace bialgebraic quantum field theory** is a functor

$$F : \mathcal{C} \to \mathcal{LB}_{cc}(\mathcal{V})$$

where $\mathcal{LB}_{cc}(\mathcal{V})$ is the category of cocommutative Laplace bialgebras in $\mathcal{V}$, such that the causality and cluster decomposition conditions are satisfied.

Now, modifying the construction of the properad $\mathcal{P}^{\mathcal{C}}$ to include the causality and cluster decomposition conditions, we have the following:

Proposition 6.5.6: Properad of Laplace Hopf AQFTs

Let $(\mathcal{C}, \perp)$ be an orthogonal category and $\mathcal{V}$ be a symmetric monoidal category whose tensor product distributes over finite coproducts. There is a properad $\mathcal{P}_H^{\mathcal{C}}$ whose algebras are the cocommutative Laplace bialgebraic AQFTs in $\mathcal{V}$.

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